95-10

MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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CORRIGENDA.

Vol. XI.

Page vi., *insert* Question 2003 .... p. 106.
„ viii., *omit* Question 2603 .... p. 105.
„ x., line 4, *omit* the factor 4.
„ x., *insert* Question 2730 .... p. 107.
„ xiii., *omit* Question 2782 .... p. 107.
„ xv., Question 2875, *for* p. 106 *read* p. 105.
„ 25, line 17, should be as follows :—

$$\frac{3\pi}{ab^3}\int_0^{\frac{1}{2}a}(b\beta^2-\beta^3)\,d\alpha=\frac{6\pi}{b^5}\int_b^{\frac{1}{2}b}(b\beta^2-\beta^3)\,(b-2\beta)\,d\beta$$

„ 58, in the Figure, the dotted curve should *touch* the parabola at C.
„ 81, line 21, *for* $\sin b \sin c$ *read* bc.
„ 94, line 20, *for* ellipse and a concentric hyperbola, &c., *read* a rectangular hyperbola and a circle whose diameter is any chord of the hyperbola (and therefore the complementary common chord a diameter of the hyperbola).
„ 97, line 16, *for* them in 4-plets *read* four in m-plets.
„ 109, line 7 from bottom, *for* $\frac{x}{a}+\frac{y}{b}$ *read* $\frac{x}{x}=\frac{y}{b}$.
„ 111, line 22, *omit* the factor 4.
„ 112, lines 1, 2, *for* $2\sin 2\theta$ *read* $\sin 2\theta$; line 3, *for* $2\cos 2\theta$ *read* $\cos 2\theta$; line 4, *for* $\sin 4\theta$ *read* $\frac{1}{2}\sin 4\theta$, and *for* $4\sin^2 2\theta$ *read* $\sin^2 2\theta$; line 5, *for* $4\cos^2 2\theta$ *read* $\cos^2 2\theta$, and *for* $\frac{1}{4}\cos^2 2\theta$ *read* $\frac{1}{4}\sin^2 2\theta$; lines 6, 9, 25, *omit* the factor 4; line 10, *for* $4\rho^2\left(\frac{dz}{d\theta}\right)$ *read* $\rho^2\left(\frac{dz}{d\theta}\right)^2$; line 2, *for* $\frac{\delta\theta^2}{z}$ *read* $\frac{\delta\theta^2}{2}$; line 22, *for* $-\lambda$ *read* -2λ.

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„ 52, line 11, *for* $\frac{1}{ar^2}$ *read* $\frac{2}{ar^2}$; line 12, *omit* the $-$ before $\int$; line 13, *for* $\frac{1}{3ar^2}$ *read* $\frac{2}{3ar^2}$.
„ 58, line 25, *for* p. 28 *read* p. 30.
„ 71, line 7, *for* (1) *read* (2), and *for* (2) *read* (3).

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Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2802. (Proposed by Professor SYLVESTER.)—Prove that, out of a set of numbers consisting of the first n coefficients in $(1+x)^{2n-1}$, or of the first n coefficients and the half of the middle coefficient in $(1+x)^{2n}$, terms may be selected, taken some positively and others negatively, so as to produce the sum 2^{n-1}. For example, if $n=6$, out of the set 1, 11, 55, 165, 330, 462 we may extract
$$-462+330+165-1 = 32,$$
and out of the set 1, 12, 66, 220, 495, 792, $\frac{1}{2}(924)$ we may extract
$$495-\tfrac{1}{2}(924)-1 = 32.$$

Solution by the Rev. J. WOLSTENHOLME, M.A.

If $a_0, a_1, a_2, \ldots$ be the coefficients of the expansion of $(1+x)^{2n-1}$, and $i=\sqrt{(-1)}$ as usual,
$$(1+i)^{2n-1} = a_0-a_2+a_4-\ \ldots\ +i(a_1-a_3+\ \ldots),$$
$$\therefore\ (1+i)^{2n-1}+(1-i)^{2n-1} = 2(a_0-a_2+a_4-\ldots) = 2^{\frac{2n-1}{2}}\left(2\cos(2n-1)\frac{\pi}{4}\right),$$
$$\therefore\ a_0-a_2+a_4-a_6+\ \ldots\ = 2^{n-1}\left(\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}\right) = \pm 2^{n-1};$$
and any of the coefficients a_{n-1+r} is equal to the corresponding coefficient a_{n-r}; so that from the first n coefficients of the expansion of $(1+x)^{2n-1}$ terms may be selected, taken some positively and others negatively, so as to produce 2^{n-1}. So if $(1+x)^{2n} = a_0+a_1x+\ \ldots\ +a_rx^r+\ \ldots$ &c.,

$$(1+i)^{2n} \pm (1-i)^{2n} = 2(a_0-a_2+a_4-\ldots) \text{ or } 2i(a_1-a_3+a_5-\ldots)$$
$$= 2^n\left(2\cos\frac{n\pi}{2}\right) \text{ or } -2^n\left(2i\sin\frac{n\pi}{2}\right),$$

$$\therefore\ 2^n\cos\frac{n\pi}{2} = a_0-a_2+a_4-\ldots,\quad 2^n\sin\frac{n\pi}{2} = -(a_1-a_3+a_5-\ldots),$$

$$\therefore\ 2^n\left(\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right) = a_0+a_1-a_2-a_3+a_4+a_5-a_6-\ldots.$$

In the right-hand side, substituting a_0 for a_{2n}, a_1 for a_{2n-1}, a_2 for a_{2n-2}, &c., it is seen that some destroy each other, and some repeat; but a_n, the coefficient of the middle term, appears only once, either positively or negatively. Hence $2^{n-1}\left(\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right)$, or $\pm 2^{n-1}$ can be formed by selecting terms from the first n coefficients and half the middle coefficient of $(1+x)^{2n}$, to be taken either positively or negatively.

2884. (Proposed by MATTHEW COLLINS, B.A.)—Subtract each of the seven Dominical Letters A, B, C, D, E, F, G from the sum of the other six of them, and then find the coefficient of the term ABCDEFG in the continued product of the seven remainders thus obtained.

Solution by the REV. J. L. KITCHIN, M.A.

Take, for generality, n letters, and put S for their sum; then the factors similar to those in the question are

$$S-2A,\quad S-2B,\quad S-2C,\quad \ldots.,\quad S-2L;$$

and the product is easily seen to be

$$S^n-2(A+B+C+\ldots+L)S^{n-1}+2^2(A.B+A.C+B.C+\ldots)S^{n-2}-\ldots$$
$$-2^3(A.B.C+A.C.D+B.C.D+\ldots+)S^{n-3}+\ldots+(-2)^n A.B.C\ldots L.$$

Now the term in S^n containing ABC...L has for coefficient, by the multinomial theorem, $\dfrac{\lfloor n}{\lfloor 1\ \lfloor 1} = \lfloor n$.

Similarly, the coefficient of BC ... L in $S^{n-1} = \lfloor n-1$,

and the coefficient of AC ... L ,, $= \lfloor n-1$,

&c. &c. = &c.

And there are n such sets, each multiplied by A, B, &c., in order; in all $= \lfloor n$.

In S^{n-2}, the coefficient of CD ... L $= \lfloor n-2$,

&c. &c. = &c.,

and there are $\frac{n(n-1)}{1.2}$ such sets corresponding in order with the terms in $AB + AC + \ldots$

therefore coefficient of ABC...L in this set $= \frac{\lfloor n}{1.2}$, and so on;

therefore coefficient of ABC...L

$$= \lfloor n \left\{1 - 2 + \frac{2^2}{1.2} - \frac{2^3}{1.2.3} + \cdots + \frac{(-2)^n}{\lfloor n}\right\} = \lfloor n \cdot \epsilon_n^{-2},$$

adopting the notation lately introduced in this Journal.

If $n = 7$, the coefficient is easily found to be 656.

2841. (Proposed by A. Martin.)—A given cylindrical vessel, filled with water, is placed with its base upon a horizontal plane. It is required to find the angle of inclination to which the plane must be raised before the vessel will fall, the water being at liberty to overflow the top. The base is supposed to be fixed so as to prevent it from sliding, but not from tilting, when the plane is inclined.

Solution by J. J. Walker, M.A.

Let h be the height of the cylinder, a the radius of base, θ the inclination at which tilting takes place, x, y the distances of the centre of gravity of the fluid remaining in the cylinder from axis and base. Then the condition

$$x + y \tan\theta = a \ldots\ldots\ldots\ldots\ldots (1)$$

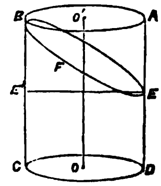
evidently expresses that the vertical through the centre of gravity meets the circumference of the base. It will be found that

$$x = \frac{a^2 \tan\theta}{4(h - a\tan\theta)}, \quad y = \frac{4h^2 - 8ah\tan\theta + 5a^2\tan^2\theta}{8(h - a\tan\theta)},$$

whence (1) becomes

$$5a^2\tan^3\theta - 8ah\tan^2\theta + (10a^2 + 4h^2)\tan\theta - 8ah = 0.$$

If $h = 2a$, this equation will have a single real root lying between $\tan\theta = 1$ and $\tan\theta = 0{\cdot}9$.



To prove the above results, let ABCD be a cylinder, and let any ungula be cut off from it by a plane BFE making an angle θ with the base. The volume of the cylindrical wedge BE′E (its base being parallel to that of the cylinder) is equal to $\pi a^3 \tan\theta$. Its moment with respect to a plane through the axis OO′, perpendicular to EE′, is equal to $\frac{1}{4}\pi a^4 \tan\theta$. Its moment with respect to its base is equal to $\frac{5}{4}\pi a^4 \tan^2\theta$. From these the distances of the centre of gravity of the ungula from the axis and base of the cylinder are readily found to have the values given above.

2829. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Through the fourth point of intersection of two conics circumscribing the triangle of reference (whose equations are $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, $l'\beta\gamma+m'\gamma\alpha+n'\alpha\beta=0$ is drawn a straight line meeting the curves again in P, P′: prove that the tangents at P, P′ intersect on the tri-cusped quartic

$$\left\{\beta\gamma\left(\frac{1}{mn'}-\frac{1}{m'n}\right)\right\}^{\frac{1}{3}}+\left\{\gamma\alpha\left(\frac{1}{nl'}-\frac{1}{n'l}\right)\right\}^{\frac{1}{3}}+\left\{\alpha\beta\left(\frac{1}{lm'}-\frac{1}{l'm}\right)\right\}^{\frac{1}{3}}=0.$$

I. *Solution by* STEPHEN WATSON.

Put $\quad L=mn'-m'n,\quad M=nl'-n'l,\quad N=lm'-l'm,$

$$L'=\frac{1}{mn'}-\frac{1}{m'n},\quad M'=\frac{1}{nl'}-\frac{1}{n'l},\quad N'=\frac{1}{lm'}-\frac{1}{l'm};$$

then the given conics may be written

$$n\alpha\,(M\beta-N\gamma)=l\gamma\,(L\alpha-M\beta)\;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$n'\alpha\,(M\beta-N\gamma)=l'\gamma\,(L\alpha-M\beta)\;\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and a line through their fourth point is

$$M\beta-N\gamma=k\,(L\alpha-N\beta);$$

hence the point P is determined by

$$\frac{k\alpha}{l}=-\frac{(k+1)\beta}{m}=\frac{\gamma}{n},$$

and the tangent at P is $\dfrac{k^2\alpha}{l}+\dfrac{(k+1)^2\beta}{m}+\dfrac{\gamma}{n}=0\;\ldots\ldots\ldots\ldots\ldots\ldots(3).$

In like manner the tangent at P′ is

$$\frac{k^2\alpha}{l'}+\frac{(k+1)^2\beta}{m'}+\frac{\gamma}{n'}=0\;\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminate first k^2 and then $(k+1)^2$ from (3) and (4), the results give

$$k+1=\pm\left(\frac{M'\gamma}{N'\beta}\right)^{\frac{1}{2}}\quad\text{and}\quad k=\pm\left(\frac{L'\gamma}{N'\alpha}\right)^{\frac{1}{2}}\;\ldots\ldots\ldots(5),$$

and the final result may be written

$$(L'\beta\gamma)^{\frac{1}{2}}+(M'\gamma\alpha)^{\frac{1}{2}}+(N'\alpha\beta)^{\frac{1}{2}}=0,$$

as it is plain the signs must be so taken in (5) as to render this final result uniform.

II. *Solution by* R. TUCKER, M.A.

Through the fourth point of intersection O $(\alpha'\beta'\gamma')$ draw the straight line OPP′ cutting the conics in P$(\alpha_1\beta_1\gamma_1)$ and P′$(\alpha_2\beta_2\gamma_2)$; then we have the equations

$$\frac{l\alpha}{\alpha'\alpha_1}+\frac{m\beta}{\beta'\beta_1}+\frac{n\gamma}{\gamma'\gamma_1}=0,\quad \frac{l'\alpha}{\alpha'\alpha_2}+\frac{m'\beta}{\beta'\beta_2}+\frac{n'\gamma}{\gamma'\gamma_2}=0;$$

and from the identity of these lines we get

$$\frac{l\alpha_2}{l'\alpha_1}=\frac{m\beta_2}{m'\beta_1}=\frac{n\gamma_2}{n'\gamma_1}\;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The tangents at P and P′ are given by the equations

$$\frac{l\alpha}{\alpha_1^2}+\frac{m\beta}{\beta_1^2}+\frac{n\gamma}{\gamma_1^2}=0\ \ldots\ldots\ (2),\qquad \frac{l'\alpha}{\alpha_2^2}+\frac{m'\beta}{\beta_2^2}+\frac{n'\gamma}{\gamma_2^2}=0\ \ldots\ldots\ (3);$$

this latter, by virtue of (1), becomes $\dfrac{l^2\alpha}{l'\alpha_1^2}+\dfrac{m^2\beta}{m'\beta_1^2}+\dfrac{n^2\gamma}{n'\gamma_1^2}=0$ .... (4);

and we have further $\qquad \dfrac{l}{\alpha_1}+\dfrac{m}{\beta_1}+\dfrac{n}{\gamma_1}=0$ .... (5).

Eliminating $\alpha_1, \beta_1, \gamma_1$ between (2), (4), and (5), we obtain the required equation,

since $\qquad \dfrac{l'}{l}\cdot\dfrac{\alpha_1^2}{\alpha}(m'n-mn')=\dfrac{m'}{m}\cdot\dfrac{\beta_1^2}{\beta}(ln'-nl')=\dfrac{n'}{n}\cdot\dfrac{\gamma_1^2}{\gamma}(l'm-lm').$

2608. (Proposed by S. Roberts, M.A.)—Show that if (a_1, a_2) are points of contact of tangents from a point A on a cubic curve, and (b_1, b_2) similar points of contact of tangents from B, another point on the same curve; and if (a_1, b_1) and (a_2, b_2) respectively are collinear with the point of contact of a tangent from the point where the line AB meets the curve again, then A B $a_1\,a_2\,b_1\,b_2$ lie in the same conic.

I. *Solution by* T. Cotterill, M.A.

Generally, if (a_1, b_1), (a_2, b_2) are points on a cubic such that the lines through (a_1, b_1) and (a_2, b_2) meet in c on the curve, then the opposite of the four points (a_1, b_1), (a_2, b_2) is C, the tangential of c; or in other words, a conic can be drawn through (a_1, b_1), (a_2, b_2) and the two remaining points of intersection of any line through C with the cubic. But in the question, A and B are two such points, because C, A, B are the tangentials of three points c, a_1, b_1 or c, a_2, b_2, either triad being collinear points on the cubic.

II. *Solution by* W. S. McCay, B.A.

It is shown in Salmon's *Higher Plane Curves*, that the three intersections

$$Aa_1 \text{ and } Bb_2 \ldots (1),\quad Aa_2 \text{ and } Bb_1 \ldots (2),\quad a_1b_1 \text{ and } a_2b_2 \ldots (3)$$

are in one straight line. But this is the condition that the six vertices of the hexagon $Aa_1b_1Bb_2a_2$ should lie on a conic—the vertices being taken in this order, for then Pascal's theorem holds for the intersections of $\left|\begin{matrix}12 \,\&\, 45\\ 23 \,\&\, 56\\ 34 \,\&\, 61\end{matrix}\right|$.

2793. (Proposed by W. K. Clifford, B.A.)—C is the single focus of a semicubical parabola, and from any point O three tangents are drawn to the curve; if CD, CE, CF be perpendicular to them, show that DE and CF are equally inclined to the direction of the infinite branches.

Solution by J. J. WALKER, M.A.

Let $y^2 = 4ax$ be the parabola which has the given semi-cubical parabola as its evolute; then, if (ξ, η) be the point D, *i. e.* the point in which a perpendicular from focus meets the normal to the parabola at (x', y'), $\xi = x' + a$, $\eta = \frac{1}{2}y'$. Similarly for E, $\xi' = x'' + a$, $\eta' = \frac{1}{2}y''$; so that the tangent of the angle which the line DE makes with the specified direction is

$$\frac{\xi - \xi'}{\eta - \eta'} = \frac{2(x' - x'')}{y' - y''} = \frac{y' + y''}{2a}.$$

Let O be (X, Y), then if (x, y) be one of the three points normals at which meet in O, $$xy + (2a - X)y - 2aY = 0;$$
and eliminating x between this and $y^2 = 4ax$, we have a cubic in y wanting the second term. Hence, if (x''', y''') be the point at which the third normal through O meets the parabola $\frac{y' + y''}{2a} = -\frac{y'''}{2a}$, and $\frac{y'''}{2a}$ is the tangent of inclination to the specified direction of the tangent to the parabola at (x''', y'''), that is, of the line CF.

2801. (Proposed by M. JENKINS, M.A.)—Show that the number of pairs of numbers which have a given number G for their G. C. M., and another given number L (of course a multiple of G) for their L. C. M., is 2^{n-1}, where n is the number of prime bases the product of whose powers is equal to $\left(\frac{L}{G}\right)$.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

Let xG, yG be a pair of numbers whose L. C. M. is L and G. C. M. is G; then x, y must be prime to each other, and $xy = \frac{L}{G}$. If, then, this latter number be $a^p b^q c^r \ldots$ where $a, b, c \ldots$ are prime, the values of x satisfying these conditions are the different terms of the expansion of $(1 + a^p)(1 + b^q)(1 + c^r) \ldots$, 2^n in number. But since the same pair of numbers are given by x and by $\frac{L}{Gx}$, the number of different pairs of numbers is 2^{n-1}.

II. *Solution by the* PROPOSER.

If Ga and Gb be any pair, a must be prime to b, and $L = Gab$, or $ab = \left(\frac{L}{G}\right)$; and the number of different pairs a, b is therefore the number of ways of separating $\left(\frac{G}{L}\right)$ into conjugate divisors which are prime to each other. This number is well known to be 2^{n-1}.

2542. (Proposed by S. ROBERTS, M.A.)—A given conic is continually touched by a similar and similarly placed conic; find the locus of a point rigidly connected with the moving conic.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Let $ax^2 + by^2 = 1$ be the fixed conic, $a(x-h)^2 + b(y-k)^2 = m$ the moving one, $X = h + \alpha$, $Y = k + \beta$ the coordinates of a point rigidly connected with the moving conic. For tangency,

$$4(ah^2 + bk^2) = (1 + ah^2 + bk^2 - m)^2; \quad \text{or} \quad ah^2 + bk^2 = 1 + m \pm 2\sqrt{m};$$

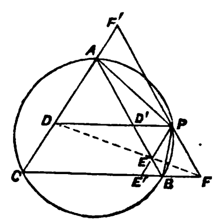
and the locus of the point is the two conics

$$a(X-\alpha)^2 + b(Y-\beta)^2 = (1 \pm \sqrt{m})^2,$$

clearly as the contact is internal or external.

2849. (Proposed by R. TUCKER, M.A.)—An ellipse is described about a triangle having the *centroid* for its centre; show that straight lines from any point on it parallel to the sides cut the sides in two sets of three points which are collinear.

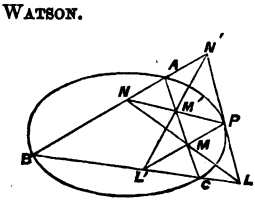
I. *Solution by* J. J. WALKER, M.A.



The triangle being concentroidal with the ellipse, the tangents at vertices will be parallel to the opposite sides; hence, if the ellipse be projected orthogonally into a circle, the triangle will be projected into an inscribed equiangular triangle.

To prove the property stated in the projected figure: let ABC be an equiangular triangle, and P any point on the circumscribing circle; and let parallels to BC, AC, AB, drawn from P, meet CA, AB, BC in D, E, F, and AB, BC, CA in D′, E′, F′. Join DE, EF, PA, PB. Then the points A, P, E, D lie on a circle; therefore ∠ PED + ∠ PAD = 2 right angles; whence ∠ PED = ∠ PBC. Again, the points P, F, B, E lie on a circle, whence ∠ PBC = ∠ FEE′; therefore ∠ FEE′ = ∠ PED, and DEF is a straight line. Similarly it may be shown that D′, E′, F′ are collinear.

II. *Solution by* STEPHEN WATSON.



Let ABC be the triangle; take BC $(=a)$ and BA $(=c)$ for axes, and denote any point P by (x, y). Then, if through F lines LPN′, PML′, PM′N be drawn parallel respectively to AC, AB, BC, cutting the sides as in the diagram, the coordinates of L, M, N are

$$\left(\frac{cx + ay}{c}, 0\right), \quad \left(x, \frac{c(a-x)}{a}\right), \text{ and } (0, y);$$

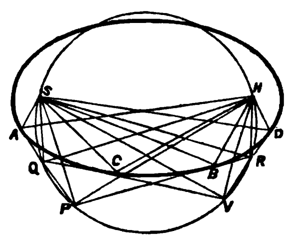
and the condition that those three points lie in a straight line is

$$c^2x^2 + a^2y^2 + acxy - ac(cx + ay) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly, that L′, M′, N′ lie in a straight line, gives the same result; moreover, (1) is an ellipse having its centre at the point $(\frac{1}{3}a, \frac{1}{3}c)$, or the centroid of the triangle ABC; hence the property stated in the question is established.

2840. (Proposed by F. D. THOMSON, M.A.)—A circle is described through the foci of a conic, and from a point P on it tangents PA, PB are drawn to the curve, meeting the circle again in Q and R. Show that the other tangents to the conic from Q and R meet on the circle.

Solution by the PROPOSER.



1. We have to show that, if V be the point where the tangents meet, the angle QSV is equal to the angle QHV.

Now, with the notation of the figure, remembering that a pair of tangents subtend equal angles at either focus,

$$\begin{aligned} QSV &= QSC + CSV = QSA + DSV \\ &= PSA - PSQ + DSR + RSV \\ &= PSB - PHQ + BSR + RSV \\ &= PSR - PHQ + RSV \\ &= PHR - PHQ + RSV. \end{aligned}$$

Also $QHV = QHC + CHV = QHA + DHV = PHA - PHQ + DHR + RHV$
$= PHB - PHQ + BHR + RHV = PHR - PHQ + RHV;$

therefore $\quad QSV - QHV = RSV - RHV,$

therefore $\quad QSV - RSV = QHV - RHV.$

But $\quad QSV + RSV = QSR = QHR = QHV + RHV;$

therefore $\quad QSV = QHV.$

2. The above is a particular case of the general theorem, that if two conics are such that a quadrilateral may be inscribed in one and circumscribed about the other, then an infinite number of such quadrilaterals may be found.

In this case, the conic touches the quadrilateral formed by the tangents through the foci, and the circle passes through the foci and the circular points, that is, circumscribes the same quadrilateral.

The condition that two conics may be so related, may be obtained by forming the invariants of the two conics referred to the triangle formed by the diagonals. Their equations are

$$(x+y+z)(y+z-x) + \kappa(z+x-y)(x+y-z) = 0 \ldots\ldots\ldots\ldots (1),$$

and

$$ax^2 + by^2 + cz^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

with the condition that (2) touches the sides of the quadrilateral, *i. e.*

$$bc + ca + ab = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

(1) may be written $\left(1, \; -1, \; -1, \; \frac{\kappa+1}{\kappa-1}, \; 0, \; 0 \between x, y, z\right)^2 = 0,$

of which the tangential equation is

$$\left(1-\left(\frac{\kappa+1}{\kappa-1}\right)^2, \; -1, \; -1, \; -\frac{\kappa+1}{\kappa-1}, \; 0, \; 0 \between \lambda, \mu, \nu\right)^2 = 0.$$

(2) has for its tangential equation

$$\left(bc, \; ca, \; ab, \; 0, \; 0, \; 0 \between \lambda\mu\nu\right)^2 = 0.$$

Therefore, with the notation of Salmon's *Conics*,

$$\Delta = 1-\left(\frac{\kappa+1}{\kappa-1}\right)^2, \qquad \Delta' = abc,$$

$$\Theta = a\left[1-\left(\frac{\kappa+1}{\kappa-1}\right)^2\right]-b-c = a\Delta-b-c, \qquad \Theta' = bc-ca-ab.$$

But, from (3), $a = -\frac{bc}{b+c}$; hence we have

$$\Delta' = -\frac{b^2c^2}{b+c}, \qquad \Theta = -\frac{bc}{b+c}\Delta-(b+c), \qquad \Theta' = 2bc;$$

therefore $$\Theta\Theta' - 2\Delta'\Delta - \frac{\Theta'^3}{4\Delta'} = 0,$$

a homogeneous relation between the invariants, and therefore the condition required.

3. If the points P and Q coincide, PA is a common tangent, and the points V and R will coincide, and therefore VD will be another common tangent. Hence the line joining the points of contact of common tangents touches the conic, and therefore by symmetry is the tangent at the end of the minor axis. This gives a solution of Mr. WOLSTENHOLME's Question 2744.

II. *Solution by* R. TUCKER, M.A.

From any point P on the focal circle draw a pair of tangents to the conic, and from one of the points of section with the circle draw a third tangent cutting the circle in Q, and join Q with the remaining point of section, and produce the line. If now $Y_1, Y_2, Y_3, Y_4, Z_1, Z_2, Z_3, Z_4$ be the feet of perpendiculars from S and H on the three tangents and the line through Q, we have

$$SY_1 \cdot SY_3 = \pm SY_2 \cdot SY_4, \qquad HZ_1 \cdot HZ_3 = \pm HZ_2 \cdot HZ_4.$$

Multiplying these equations, and remembering that $SY \cdot HZ = b^2$, we have

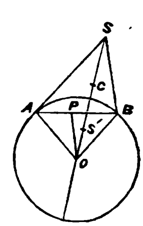
$$b^4 = b^2 \cdot SY_4 \cdot HZ_4, \quad i.e., \quad SY_4 \cdot HZ_4 = b^2;$$

hence, as may readily be shown by the aid of this result, the line through Q is a tangent to the conic. This proves the property of the question.

NOTE.—$SY_1 \cdot SY_3 = \pm SY_2 \cdot SY_4$ by the equation $\alpha\gamma = \pm\beta\delta$ for a circle. Also, if product of perpendiculars from foci on the line $y = mx + c$ be $= b^2$, we have $c^2 - m^2a^2e^2 = b^2(1+m^2)$, therefore $c^2 = b^2 + m^2a^2$, the condition for tangency.

2684. (Proposed by MORGAN JENKINS, M.A.)—From a point S two straight lines are drawn meeting a circle in A and B, so that the rectangle SA . SB is constant; show that the envelope of the secant AB is a central conic of which S and its conjugate point are foci; and that the conic is an ellipse or hyperbola according as the given rectangle is greater or less than the rectangle contained by the intercepts made by the circle on any straight line through S.

I. *Solution by the* PROPOSER.



Let O be the centre of the circle; then, taking OS for initial line, and putting θ, θ' for the angles AOS, BOS, p for the perpendicular from O on AB, ϕ for the angle between AB and OS, OA$=c$, OS$=k$, and $l^2=$ the given rectangle, we have

$$p = c\cos\tfrac{1}{2}(\theta-\theta'),\quad \phi = \tfrac{1}{2}\pi + \tfrac{1}{2}(\theta+\theta'),$$
$$(c^2+k^2-2ck\cos\theta)(c^2+k^2-2ck\cos\theta') = l^4.$$

Eliminating θ and θ', we shall obtain

$$\left(p - \frac{c^2+k^2}{2k}\sin\phi\right)^2 + \left(\frac{c^2-k^2}{2k}\right)^2\cos^2\phi = \left(\frac{l^2}{2k}\right)^2.$$

This is the equation of a central conic referred to the major axis as initial line and a point on it as origin. The distances of the centre of the conic from the origin and from the focus are $\dfrac{c^2+k^2}{2k}$ and $\dfrac{c^2-k^2}{2k}$ respectively; whence the distances of the foci from the origin O are k and $\dfrac{c^2}{k}$ respectively; hence S and its conjugate point are the foci. Also $\dfrac{l^2}{2k}$ is the semi-major axis; hence the conic is an ellipse or hyperbola according as $c^2 \backsim k^2$ is $<$ or $> l^2$.

II. *Solution by* J. J. WALKER, M.A.

Let O be the centre of the given circle, P any point in AB; also let OA$=a$, OS$=b$, $\angle$AOS $=\phi$, $\angle$BOS $=\phi'$, $\angle$POS $=\theta$, AS . SB $=c^2$. For shortness, let $\cos\frac{1}{2}(\phi+\phi') = u$, $\cos\frac{1}{2}(\phi-\phi') = v$, $a^2+b^2 = h^2$, $2ab = h'^2$, and $(a^2-b^2)^2-c^4 = k^4$.

Then it is easily found that the polar equation to the secant AB may be thrown into the form

$$r^2\sin^2\theta\,(h'^4u^2-2h^2h'^2uv+h^4-c^4)-h^4(r\cos\theta\, v-au)^2 = 0 \;\ldots\ldots (1),$$

while, by the terms of the question, the parameters u, v are connected by the equation
$$h'^4(u^2+v^2)-2h^2h'^2uv+k^4 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Differentiating (1) and (2) with respect to u and v, and eliminating du, dv between the results, we obtain

$$a(ah^2-rh'^2\cos\theta)u^2+(r^2-a^2)h'^2uv+r(ah'^2\cos\theta-rh^2) = 0 \;\ldots\; (3).$$

Again, subtracting (1) multiplied by k^4 from (2) multiplied by $r^2\sin^2\theta\,(h^4-c^4)$,

$$(r^2h'^2\sin^2\theta+a^2k^4)u^2-2r(rh^2h'^2\sin^2\theta+ak^4\cos\theta)uv+r^2(h'^4\sin^2\theta+k^4)v^2 = 0 \;\ldots\ldots (4).$$

Finally, eliminating u, v between (3) and (4), the result may be reduced to the product of the factors

$$r^2 \sin^2\theta \left\{ \left[(a^2+r^2)\, b - (a^2+b^2)\, r\cos\theta \right]^2 + (a^2-b^2)^2 r^2 \sin^2\theta \right\},$$

and $$k^8 + 4b \left\{ (a^2+r^2)\, b - (a^2+b^2)\, r\cos\theta \right\} k^4 - 4(a^2-b^2)^2 b^2 r^2 \sin^2\theta.$$

The first factor, being independent of k, is irrelevant. The second, equated to zero, is the required envelope; which, if $2bx = 2br\cos\theta - (a^2+b^2)$, $y = r\sin\theta$, becomes $4b^2(k^4x^2 - c^4y^2) = c^4k^4$, a hyperbola if $a^2 \backsim b^2 > k^2$, an ellipse if $a^2 \backsim b^2 < k^2$. The foci are S and its reciprocal point with respect to the circle. If $a^2 \backsim b^2 = k^2$, the secant AB passes either through S or S′. If S is on the circle, S and S′ coincide, and the ellipse becomes a circle, the solution of Quest 2100. (*Reprint*, Vol. X., p. 97.)

2747. (Proposed by W. S. McCay, B.A.)—Prove that the locus of the foot of the bifocal perpendicular on the tangent plane to a quadric is a concentric sphere whose radius is the primary axis.

Solution by the Proposer.

For the foot of this perpendicular is a point through which can be drawn three rectangular tangent planes to three confocals. Since the two tangent planes through a line to the two confocals that it touches are at right angles, the locus is a sphere. (See Salmon's *Geometry of three Dimensions*, Arts. 168, 172.)

2894. (Proposed by the Rev. J. Wolstenholme, M.A.)—An ellipse and hyperbola are confocal, and the asymptotes of the latter lie on the equi-conjugates of the former; prove that any ellipse which passes through the ends of the axes of the given ellipse will cut the hyperbola orthogonally.

Solution by W. H. H. Hudson, M.A.

If $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ be the hyperbola, the equation of an ellipse confocal and with its equi-conjugates along the asymptotes may be found to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2+b^2}{a^2-b^2}.$$

Any ellipse through the intersection of this with $xy=0$ may be taken to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2\lambda xy = \frac{a^2+b^2}{a^2-b^2}.$$

If this intersect the hyperbola in (x, y), the tangents to the two curves at the point of section are

$$\frac{Xx}{a^2}-\frac{Yy}{b^2}=1,\quad X\left(\frac{x}{a^2}+\lambda y\right)+Y\left(\frac{y}{b^2}+\lambda y\right)=\frac{a^2+b^2}{a^2-b^2}.$$

Now, since (x, y) is in both curves,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+2\lambda xy=\frac{a^2+b^2}{a^2-b^2}\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right);$$

and this will be found to be the same as the condition that the two tangents are at right angles, viz., $\frac{x}{a^2}\left(\frac{x}{a^2}+\lambda y\right)-\frac{y}{b^2}\left(\frac{y}{b^2}+\lambda x\right)=0.$

2782. (Proposed by M. W. Crofton, F.R.S.)—1. A given point is known to be within a certain circle of given radius, but unknown position; find the chance that another given point is also within the circle.

2. Three given points are known to be within a certain circle, which is otherwise altogether unknown; determine the most probable position of its centre.

3. Two given points are known to be within a circle, and a third given point is known to be outside it; determine the most probable position of its centre.

Solution by the Rev. J. Wolstenholme, M.A.

1. If A, B be the two given points, $AB=2c$, a the radius of the given circle, P its centre, then all the possible positions of P lie within the circle centre A and radius a; and in order that B may also lie within the circle, P must lie within the circle centre B and radius a; the probability is then obviously $\frac{\theta-\sin\theta\cos\theta}{\pi}$, where $\cos\theta=\frac{c}{a}$.

2. If x, y, z be the distances of the centre of the circle from three given points A, B, C, the radius of the circle must be greater than the greatest of the three x, y, z; the most probable position of the centre is then when this greatest of the three distances has its least value. If now x be the greatest of the three, through B′, C′, middle points of AC, AB, draw straight lines at right angles to these sides, meeting in O the centre of the circle circumscribing ABC. Then, in order that x may be $>y$ or z, the centre of the circle must lie in the angle vertically opposite B′OC′, and the least value will be when the centre is at O, if the angles B, C be acute. So if y, or z, be the greatest. Hence the most probable position of the centre, when the triangle is acute-angled, is O. If, however, one of the angles (A) be obtuse, the least value of x, when x is the greatest of the three, will be when the centre is at O, but the least value of y, (or z), when it is the greatest, will be when the centre is at A′, the middle point of the side opposite the obtuse angle, and since this least value is less than the former, A′ is the most probable position of the centre.

3. If A be the point without the circle, then if r be the radius, $x>r$, $y<r$,

$z < r$; hence the centre must lie in the angle vertically opposite B′OC′, and the most probable position of the centre is that in which, if y be the greater of the two y, z, $x-y$ is the greatest possible, which, if OA′ be drawn at right angles to BC and the angle C be obtuse, will be found when a hyperbola with foci A, B touches OA′ with the branch to which B is an interior focus. If the angles be acute, it seems to me the most probable position must be at infinity on OA′.

2864. (Proposed by Professor SYLVESTER.)—A body moves in an ellipse with *uniform* velocity under the attraction of two central forces at the foci. Show that the forces to each focus are always equal, and vary inversely as the product of the focal distances.

Solution by J. J. WALKER, M.A.; JAMES DALE; *and others.*

Since the velocity is uniform, the resultant of the attractive forces must at every point be normal to the ellipse; therefore their tangential components must be equal and opposite; and the lines to the foci making equal angles with the tangent at the point, it follows that the forces to the two foci must be equal. Again, the normal force at each point must vary as the curvature, and therefore the force to either focus as the curvature directly, and the cosine of angle between normal and line to focus inversely; *i. e.*, in the sesquiplicate ratio of product of the focal distances inversely, and the subduplicate ratio of the same product directly, or inversely as that product.

2811. (Proposed by M. JENKINS, M.A.)—If

$$u = \sum_{s=0}^{s=p-2} \frac{\Pi(m+n)r}{\Pi\left\{mr+k(p-1)\right\}\,\Pi\left\{nr-k(p-1)\right\}} \equiv c\,(\text{mod.}\,p),$$

where p is any odd prime number; m, n, and r any positive integers; and k takes all integral values between $\dfrac{nr}{p-1}$ and $-\dfrac{mr}{p-1}$, inclusive of those quantities, if integral; show that c has the same value for all values of r which have the same G. C. M. with $(p-1)$.

Solution by the PROPOSER.

In the series $v = \sum_{s=0}^{s=p-2} \theta^{(m+n)\,\text{Ind}_\gamma(\gamma^s+1)-ms}$, where γ is any primitive root of p, and $\text{Ind}_b\,a$ means the congruential logarithm of a to the base b, that is, $\text{Ind}_b\,a = x$ if $b^x \equiv a\,(\text{mod.}\,p)$, for θ put γ^r; and we obtain

$$\sum_{s=0}^{s=p-2} \frac{(\gamma^s+1)^{(m+n)r}}{\gamma^{mrs}} = \omega \text{ suppose.}$$

Expanding the numerator of ω and dividing by the denominator, we obtain a series of terms of the form $C\gamma^{\rho s}$; and $\Sigma_{s=0}^{s=p-2}\gamma^{\rho s} \equiv 0$ (mod. p) except when $\rho=0$ or a multiple of $(p-1)$, in which case

$$\Sigma_{s=0}^{s=p-2}\gamma^{\rho s} \equiv -1 \text{ (mod. } p).$$

Hence, collecting these terms, we find that $\omega \equiv -u$ (mod. p), a result which is independent of the particular primitive root employed.

Since γ^r can be made $\equiv \beta^q$ or α^d, where d is the G. C. M. of r and $(p-1)$, q is any other integer having d for the G. C. M. of itself and $(p-1)$, and α, β are certain other primitive roots of p, the above-stated theorem follows.

I will add the following theorem:—If $(m+n)r$ be, and mr (or nr) be not, a multiple of $(p-1)$, then $u \equiv \pm 1$ mod. p according as mr is even or odd.

For the numerator in $\omega \equiv 1$ (mod. p) for all values of s except $\frac{p-1}{2}$; hence

$$\omega \equiv \left\{\Sigma_{s=0}^{s=p-2}\frac{1}{\gamma^{mrs}}\right\} - \frac{1}{\gamma^{mr\frac{p-1}{2}}} \equiv -\frac{1}{\gamma^{mr\frac{p-1}{2}}},$$

and $t \equiv \dfrac{1}{\gamma^{mr\frac{p-1}{2}}} \equiv \pm 1$ according as mr is even or odd.

2603. (Proposed by T. COTTERILL, M.A.)—In a plane, (1) on every line, there are two points harmonic conjugates to the conics through four points ABCD. They are real, if the product of the four triangles ABC, BCD, CDA, DAB, and the perpendiculars from the points ABCD on the line, is positive. If a line does not cut a conic, why are the conjugates on the line real, to any set of four points on the conic? Give a rule for determining the nature of the points on a line, when the product of the triangles is (1) positive, (2) negative.

(2.) Through any point, there are two lines conjugate to the conics inscribed in four lines. They are real in those regions of the plane formed by the four lines in which the product of perpendiculars from the point has the same sign as that of points in the convex quadrilateral. Hence every point on or within a conic inscribed in the four lines lies in these regions.

Solution by the REV. J. WOLSTENHOLME, M.A.

The series of conics described through four given points cut any straight line in points in involution. The foci of this involution are, of course, the points of contact of the two conics through the four points touching the line; and these two points will therefore be conjugates with respect to any conic through the four points. The conditions of their reality are, (1) when the four points form a convex quadrangle, the straight line must be so situated as to have either all four points on the same side of it, or two on one side and two on the other; (2) when the four points form a reentrant

quadrangle, one of them must be on the opposite side of the straight line to the other three.

Of course, reciprocally, the tangents drawn from a point to a series of conics inscribed in a quadilateral form a pencil in involution; whose double rays, the tangents to the two conics which pass through the point, are conjugates with respect to any conic of the series. These will, therefore, be real if the point lie in the convex quadrilateral formed by the lines, or in any of the portions of space into which a point can pass from this by crossing two of the lines simultaneously.

I am not sure that I understand Mr. Cotterill's query. But, given a straight line and a conic which does not meet it in real points, it is manifest that whatever four points may be taken on the conic will either, in an ellipse and parabola, form a convex quadrangle and be all on one side of the straight line; or, in a hyperbola, whose branches must lie on opposite sides of the straight line, will form a convex quadrangle, and be all on one branch, or two on one branch and two on the other; or will form a reentrant quadrangle, and thus be three on one branch and one on the other. In all such cases, then, the conjugate points on the line, to all conics through the four points, will be real.

It may be noticed that in the first case the three pairs of straight lines through the four points are particular cases of the series, and cut the given straight lines in six points in involution whose foci are therefore readily found. Correspondingly, in the second case, the straight lines joining the given points to the ends of the diagonals of the quadrilateral form a pencil of six rays in involution and the double rays are known.

In considering this question, I discovered the property proposed for proof in Question 2875.

2735. (Proposed by R. TUCKER, M.A.)—The escribed radii of a triangle are in harmonical progression, and the mean escribed radius is a geometrical mean between the inscribed and circumscribed radii: show that the cosine of the mean angle of the triangle is $\frac{8}{9}$, and that the circumscribed radius is nine times the inscribed radius.

Solution by the PROPOSER.

Since the escribed radii are in harmonic progression, we have

$$a+c = 2b, \quad \text{and} \quad 2s = 3b;$$

also
$$\left(\frac{2\Delta}{b}\right)^2 = \mathrm{R}r = \frac{abc}{4\Delta}\cdot\frac{\Delta}{s} = \frac{abc}{6b} = \frac{ac}{6};$$

therefore
$$a^2 \sin^2 \mathrm{C} = \tfrac{1}{6} ac, \quad \text{or} \quad \sin \mathrm{A} \sin \mathrm{C} = \tfrac{1}{6}.$$

Now
$$1+\cos \mathrm{B} = \frac{(a+c)^2-b^2}{2ac} = \frac{3b^2}{2ac} = 9\sin^2 \mathrm{B}, \quad \text{hence} \quad \cos \mathrm{B} = \tfrac{8}{9}.$$

Also
$$\frac{\mathrm{R}}{r} = \frac{bs}{2\sin \mathrm{B}\,.\,\Delta} = \frac{3b^2}{2ac\sin^2 \mathrm{B}} = 9, \quad \text{therefore} \quad \mathrm{R} = 9r.$$

We also get
$$\tan \tfrac{1}{2}a \,.\, \tan \tfrac{1}{2}\mathrm{C} = \tfrac{1}{3}.$$

2582. (Proposed by W. S. BURNSIDE, M.A.)—Supposing that two quadrics have double contact, their axes remaining parallel; then (1) determine the *complete* locus of the centre of one quadric relative to the other; and (2) find the surfaces traced out by their chords of contact.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

If two conicoids have double contact, they intersect in two plane curves real or impossible. Let these two be

$$ax^2 + by^2 + cz^2 = 1, \quad A(x-X)^2 + B(y-Y)^2 + C(z-Z)^2 = 1,$$

then $$ax^2 + by^2 + cz^2 - 1 + \lambda\left\{A(x-X)^2 + \dots - 1\right\}$$

must be the product of two linear factors.

Hence $$-\lambda = \frac{a}{A}, \text{ or } \frac{b}{B}, \text{ or } \frac{c}{C}: \text{ suppose } \frac{a}{A} \quad \dots\dots\dots\dots (1),$$

then $$X = 0, \ (b+\lambda B)(c+\lambda C)\left\{\lambda(BY^2 + CZ^2 - 1) - 1\right\}$$
$$= \lambda^2 C^2 Z^2 (b+\lambda B) + \lambda^2 B^2 Y^2 (c + \lambda C) \dots\dots (2),$$

or $$X = 0, \quad \frac{BbY^2}{Ab - Ba} + \frac{CcZ^2}{Ac - Ca} = \frac{1}{A} - \frac{1}{a};$$

and there will be solutions giving two conics in the other principal planes, but these need not all be real.

II. *Solution by the* PROPOSER.

1. Let the equations of the quadrics be written in the forms

$$U \equiv ax^2 + by^2 + cz^2 - 1 = 0,$$
$$V \equiv a'(x-\alpha)^2 + b'(y-\beta)^2 + c'(z-\gamma)^2 - 1 = 0;$$

then, since they have double contact, $V + \lambda U \equiv LM$, whence the discriminant of $V + \lambda U$ vanishes, giving

$$\frac{aa'}{\lambda a + a'}\alpha^2 + \frac{bb'}{\lambda b + b'}\beta^2 + \frac{cc'}{\lambda c + c'}\gamma^2 = 1 + \frac{1}{\lambda}:$$

also $$(\lambda a + a')(\lambda b + b')(\lambda c + c') = 0,$$

as the section of $V + \lambda U$ by a plane is two lines.

Taking one of the roots of the latter equation, and substituting it for λ in the former, *sc.* $\lambda = -\dfrac{c'}{c}$, we find

$$\frac{aa'.\alpha^2}{a'c - ac'} + \frac{bb'.\beta^2}{b'c + bc'} = \frac{1}{c} - \frac{1}{c'} \quad \text{and} \quad \gamma = 0$$

as the equations of a conic on which the centre must lie. Similarly we get two other conics (all of which may not be real).

2. It is easily seen that the chord of contact generates cylinders perpendicular to the principal planes; and if the coordinates of the centre be (α, β), the chord of contact passes through the point (x, y) given by the equations $\alpha = (\lambda a' + a)x$, $\beta = (\lambda b' + b)y$, $0 = \lambda c' + c$; whence from the three conics we derive three cylinders.

2772. (Proposed by R. TUCKER, M.A.)—The curves

$$y^2 = 4ax \ldots (1), \quad y = a\left(\epsilon^{\frac{x}{2a}} + \epsilon^{-\frac{x}{2a}}\right) \ldots (2), \quad \text{and} \quad \sin\frac{y}{2a} = \epsilon^{\frac{x}{2a}} \ldots (3),$$

are referred to the same origin and axes; show that, at the points for which a tangent to (2) is at right angles to tangents to (1) and (3), the radii of curvature are in continued proportion.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

The intrinsic equations of the three are

$$(1)\ \frac{ds}{d\phi} = \frac{2a}{\sin^3\phi}, \quad (2)\ \frac{ds}{d\phi} = \frac{2a}{\cos^2\phi}, \quad (3)\ \frac{ds}{d\phi} = \frac{2a}{\sin\phi},$$

ϕ being the angle made by the tangent with the axis of x, from which the result is obvious.

II. *Solution by the* PROPOSER.

By reference to my article on Radials (*Reprint*, Vol. I., §§ 10, 13, 22), it will be seen that the radials for (1), (2), (3) are, respectively,

$$r\cos^3\theta = 2a, \quad r\sin^2\theta = 2a, \quad r\cos\theta = 2a.$$

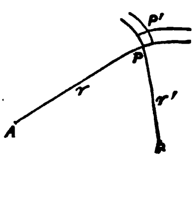
Hence, if ρ_1, ρ_2, ρ_3 be the radii of curvature supposed in the question, we have

$$\rho_1\rho_3 = 4a^2\sec^4\theta = (2a\sec^2\theta)^2 = \rho_2^2,$$

since in (2) we have to write $\frac{1}{2}\pi - \theta$ for θ.

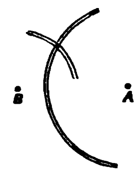
2398. (Proposed by M. W. CROFTON, F.R.S.)—A point P is laid down in a plane by taking random values for AP and BP, its distances from two fixed points A, B; and the plane is covered with an infinite multitude of such points. Show that their distribution will be such that the circle on AB as diameter will be the line of maximum density; and that all circles through A, B are lines of equal density—the density being infinitely small along the line AB.

I. *Solution by the* PROPOSER.



Draw circles from A, B as centres through any point P; radii r, r'; and then two consecutive circles, radii $r+\delta r$, $r'+\delta r'$; δr, $\delta r'$ being constant increments. Then the number of the points which fall inside the elementary parallelogram PP′ is constant. Hence the density at P $\propto \frac{1}{PP'} \propto \frac{\sin P}{\delta r\,\delta r'} \propto \sin P$; so that the density is proportional to sin P. The rest follows.

II. *Solution by the* Rev. J. Wolstenholme, M.A.



If we take a point P situated within the elementary area included within two circles (centre A, radii r, $r+\delta r$), and two circles (centre B, radii r', $r'+\delta r'$); then, δr, $\delta r'$ being by the conditions of the problem independent of the position of P, the density about P will vary inversely as the area included by these four curves (I mean that δr, $\delta r'$ being assigned arbitrary values, the number of points of intersection included within this area will be given, and therefore the density varies inversely as the area). But if we take the equations of the circles $(x+a)^2+y^2=r^2$, $(x-a)^2+y^2=r'^2$, $(\mathrm{AB}\equiv 2a)$, this elementary area is (see Todhunter's *Integral Calculus*, Change of Variables in a Multiple Integral) $\pm\left(\frac{dx}{dr}\frac{dy}{dr'}-\frac{dy}{dr}\frac{dx}{dr'}\right)\delta r\,\delta r'$; or the density varies inversely as $\frac{dx}{dr}\frac{dy}{dr'}-\frac{dy}{dr}\frac{dx}{dr'}$, or varies directly as $\frac{y}{rr'}$. The density will therefore be constant when $\frac{y^2}{(x^2+y^2+a^2)-4a^2x^2}$ or $\frac{y^2}{(x^2+y^2-a^2)+4a^2y^2}$ is constant, and for any circle through AB, $x^2+y^2-a^2=2ky$; therefore the density at any point on such circle is constant, and is to the density on the circle whose diameter is AB as $\frac{1}{\sqrt{(a^2+k^2)}}:\frac{1}{a}$, or as $a:\sqrt{(a^2+k^2)}$; hence the density is a maximum along the circle whose diameter is AB.

2546. (Proposed by R. Tucker, M.A.)—Eliminate θ between

$$\sin\theta\left\{x\sin(60^\circ-\mathrm{A}-\theta)+y\cos(60^\circ-\mathrm{A}-\theta)\right\}+y\sin\mathrm{A}=0,$$

$$\sin(\mathrm{A}+\theta)\left\{y\cos(60^\circ+\theta)-(x-a)\sin(60^\circ+\theta)\right\}=y\sin\mathrm{A}.$$

Solution by the Proposer.

Introducing quantities λ, μ, p, q, &c. to denote expressions independent of θ, we have

$$x\cos(60^\circ-\mathrm{A}-2\theta)-y\sin(60^\circ-\mathrm{A}-2\theta)$$
$$=x\cos(60^\circ-\mathrm{A})-y\sin(60^\circ-\mathrm{A})-2y\sin\mathrm{A}=\lambda,$$

$$(x-a)\cos(60^\circ+\mathrm{A}+2\theta)+y\sin(60^\circ+\mathrm{A}+2\theta)$$
$$=(x-a)\cos(60^\circ-\mathrm{A})+y\sin(60^\circ-\mathrm{A})+2y\sin\mathrm{A}=\mu;$$

whence $\quad p\cos 2\theta+p'\sin 2\theta=\lambda,\quad q\cos 2\theta+q'\sin 2\theta=\mu,$

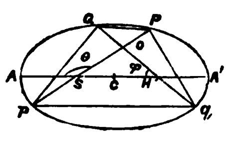
or $\quad (pq'-p'q)^2=(\lambda q'-\mu p')^2+(\mu p'-\lambda q)^2,$

where

$$p=x\cos(60^\circ-\mathrm{A})-y\sin(60^\circ-\mathrm{A}),\quad q=y\sin(60^\circ+\mathrm{A})+(x-a)\cos(60^\circ+\mathrm{A}),$$
$$p'=x\sin(60^\circ-\mathrm{A})+y\cos(60^\circ-\mathrm{A}),\quad q'=y\cos(60^\circ+\mathrm{A})-(x-a)\sin(60^\circ+\mathrm{A}).$$

2526. (Proposed by R. TUCKER, M.A.)—Through the foci of an ellipse two chords are drawn, and with them as diagonals a quadrilateral is formed; find (1) the maximum or minimum figure when they intersect at a constant angle, and (2) the locus of their point of intersection when the quadrilateral is a maximum or a minimum.

Solution by the PROPOSER.



1. Let $PQpq$ be one of the figures whose diagonals intersect in O; then area $= \frac{1}{2}Pp.Qq \sin O$.

If now $ASP = \theta$, $SOH = \alpha$, then area of

$PQpq = 2c^2 \div (1-e^2\cos^2\theta)\left\{1-e^2\cos^2(\theta-\alpha)\right\}$;

hence for a maximum or minimum,

$$1-e^2\cos^2\theta-e^2\cos^2(\theta-\alpha)+e^4\cos^2\theta\cos^2(\theta-\alpha)\dots\dots\dots\dots(\alpha)$$

must be a minimum or a maximum;

therefore $0 = \sin 2\theta+\sin 2(\theta-\alpha)-2e\sin(2\theta-\alpha)\cos\theta\cos(\theta-\alpha)$;

or, $\quad \sin(2\theta-\alpha)\cos\alpha = e^2\sin(2\theta-\alpha)\cos\theta\cos(\theta-\alpha)$;

therefore $\quad 2\theta = \alpha \text{ or } \pi+\alpha \dots\dots\dots\dots\dots\dots\dots\dots\text{(i)},$

and $2\cos\alpha = e^2\left\{\cos(2\theta-\alpha)+\cos\alpha\right\}$, i.e., $\cos(2\theta-\alpha) = \dfrac{2-e^2}{e^2}\cos\alpha\dots\text{(ii)}.$

Now if u be the variable expression (α),

$$\frac{d^2u}{d\theta^2} = (2-e^2)\cos\alpha\cos(2\theta-\alpha)-e^2\cos 2(2\theta-\alpha)\dots\dots\dots(\beta).$$

If $\quad 2\theta = \alpha, \quad (\beta) = (2-e^2)\cos\alpha-e^2$;

if $\quad \cos\alpha > \dfrac{e^2}{2-e^2}$, the quadrilateral is a maximum;

if $\quad \cos\alpha < \dfrac{e^2}{2-e^2}$, it is a minimum.

If $\quad 2\theta = \pi+\alpha, \quad (\beta) = -\left\{(2-e^2)\cos\alpha+e^2\right\}$;

this, if α is acute, gives a minimum value for the quadrilateral; if obtuse, the quadrilateral will still be a minimum until α reaches the value found by making the above vanish, and then it becomes a maximum.

For the value (ii), $\quad (\beta) = e^2-\dfrac{(2-e^2)^2}{e^2}\cos^2\alpha$;

this leads to the same results as above obtained for α, but θ will be different.

2. For the second part of the question we have

$\sin(\theta-\phi)\div\left\{1-e^2\cos^2\theta-e^2\cos^2\phi+e^4\cos^2\theta\cos^2\phi\right\}$ = a maximum or minimum;

whence, differentiating with respect to ϕ, we have

$$e^2\sin 2\phi\sin(\theta-\phi) = -\cos(\theta-\phi)(1-e^2\cos^2\phi),$$

or $\quad \cos(\theta-\phi) = e^2\cos\phi\left\{\cos\theta-\sin\phi\sin(\theta-\phi)\right\}\dots\dots\dots(\gamma).$

Let now $\angle$ HSO $= \theta'$ and SO $= \rho$, be the coordinates of O, then

$$\frac{\rho}{2ae} = \frac{\sin\phi}{\sin(\theta'+\phi)}, \quad \text{and} \quad \cot\phi = \frac{2ae-\rho\cos\theta'}{\rho\sin\theta'} \quad \ldots\ldots\ldots (\delta),$$

and (γ) becomes $\quad \sin\theta' - \cos\theta'\cot\phi = -e^2\cot\phi\left\{\cos\theta' + \dfrac{2ae\sin^2\phi}{\rho}\right\};$

therefore $\quad \cot\phi\cos\theta'(1-e^2) - \sin\theta' = \dfrac{2ae^3\sin\phi\cos\phi}{\rho}.$

Substituting from (δ), we have

$$\frac{2ae-\rho\cos\theta'}{\rho\sin\theta'}\cos\theta'(1-e^2) - \sin\theta' = \frac{2ae^3}{\rho}\cdot\frac{(2ae-\rho\cos\theta')\,\rho\sin\theta'}{(\rho^2-4ae\rho\cos\theta'+4a^2e^2)}.$$

This reduces to (changing to rectangular axes)

$$x^4(1-e^2) + x^2y^2(1-e^2) + y^4 - 6ae(1-e^2)x^3 - 6aexy^2 + 12a^2e^2(1-e^2)x^2$$
$$+ 4a^2e^2(1+e^2)y^2 - 8a^3e^3(1-e^2)x = 0.$$

2283. (Proposed by R. Tucker, M.A.)—The eccentric angles of three points P, Q, R on an ellipse being α, β, γ, find the relations these quantities satisfy when PQ, QR are (i) normals at P, Q, (ii) normals at P, R; and find the area of PQR in the two cases; also the maximum triangles.

Solution by the Proposer.

The normals at P, Q, R are given by

$$ax\sec\alpha - by\,\mathrm{cosec}\,\alpha = c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$ax\sec\beta - by\,\mathrm{cosec}\,\beta = c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$ax\sec\gamma - by\,\mathrm{cosec}\,\gamma = c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and the chords PQ, QR, RP by

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta) = \cos\tfrac{1}{2}(\alpha-\beta) \quad \ldots\ldots\ldots\ldots (4),$$
$$\frac{x}{a}\cos\tfrac{1}{2}(\beta+\gamma) + \frac{y}{b}\sin\tfrac{1}{2}(\beta+\gamma) = \cos\tfrac{1}{2}(\beta-\gamma) \quad \ldots\ldots\ldots\ldots (5),$$
$$\frac{x}{a}\cos\tfrac{1}{2}(\gamma+\alpha) + \frac{y}{b}\sin\tfrac{1}{2}(\gamma+\alpha) = \cos\tfrac{1}{2}(\alpha-\gamma) \quad \ldots\ldots\ldots\ldots (6).$$

Now, in the first case, (1) must be identical with (4), and (2) with (5); this gives us

$$\left.\begin{aligned} a^2\sec\alpha\cos\tfrac{1}{2}(\alpha-\beta) &= c^2\cos\tfrac{1}{2}(\alpha+\beta) \\ -b^2\,\mathrm{cosec}\,\alpha\cos\tfrac{1}{2}(\alpha-\beta) &= c^2\sin\tfrac{1}{2}(\alpha+\beta) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (A),$$

$$\left.\begin{aligned} a^2\sec\beta\cos\tfrac{1}{2}(\beta-\gamma) &= c^2\cos\tfrac{1}{2}(\beta+\gamma) \\ -b^2\,\mathrm{cosec}\,\beta\cos\tfrac{1}{2}(\beta-\gamma) &= c^2\sin\tfrac{1}{2}(\beta+\gamma) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (B).$$

From (A) and (B) we obtain

$$\tan\alpha\tan\tfrac{1}{2}(\alpha+\beta) = -\frac{b^2}{a^2} = \tan\beta\tan\tfrac{1}{2}(\beta+\gamma) \quad \ldots\ldots\ldots (C).$$

In the second case, we have similarly

$$\tan \alpha \tan \tfrac{1}{2}(\alpha+\beta) = -\frac{b^2}{a^2} = \tan \gamma \tan \tfrac{1}{2}(\gamma+\beta) \quad \ldots\ldots\ldots (\text{D}).$$

The equations (C) and (D) give us the required relations.

Again, the area of PQR is

$$2ab \sin \tfrac{1}{2}(\alpha-\beta) \sin \tfrac{1}{2}(\beta-\gamma) \sin \tfrac{1}{2}(\gamma-\alpha).$$

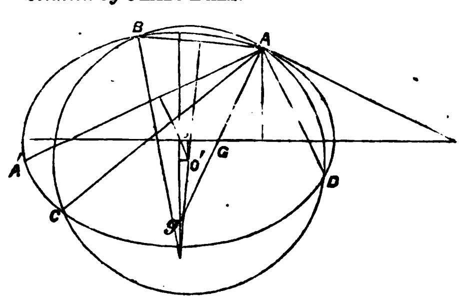
The remainder of the question is solved by taking (7) in connexion with (C) and with (D), and differentiating to get the maximum.

2803. (Proposed by Professor HIRST.)—Through every point A on a conic pass three circles which osculate the conic elsewhere, say in B, C, D. Prove that A, B, C, D lie on the circumference of a circle, and find its envelope.

2833. (Proposed by PATRICK O'CAVANAGH.)—(1.) Through every point A on a central conic pass three circles, which osculate the conic elsewhere, say in B, C, D. Prove that the diameter of the circle passing through A, B, C, D is $= \dfrac{a^2+3b^2}{2b}$ when greatest, and $\dfrac{b^2+3a^2}{2b}$ when least. (2.) Prove also that the distance between the centre of this circle, and the centre of the circle osculating the conic at B, C, or D, is when greatest $= \dfrac{3}{4}\left(\dfrac{a^2+b^2}{b}\right)$, and when least $= \dfrac{3}{4}\left(\dfrac{a^2-b^2}{a}\right)$, where a and b are the semi-axes of the conic.

2843. (Proposed by R. TUCKER, M.A.)—Two parabolas are described through the points A, B, C, D of Question 2803, show that their axes intersect on a similar concentric conic, and that their envelopes and the loci of their vertices and foci are quartic curves.

Solution by JAMES DALE.



Let (h, k) be the coordinates of a point A on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let (x, y) be another point on the ellipse such that the line $(h, k)\,(x, y)$ makes with the major axis the same angle, but in an opposite direction, as the tangent

at (x, y) does; then the equation to the line (h, k), (x, y), will be

$$k-y = \frac{b^2x}{a^2y}(h-x), \quad \text{or} \quad b^2x^2-a^2y^2-b^2hx+a^2ky = 0 \; \ldots\ldots\ldots (1)$$

which shows that the points B, C, D such that the chords common to the ellipse and the osculating circles at these points all pass through the same point h, k on the ellipse, all lie on a hyperbola which has its centre at the point $(\frac{1}{2}h, \frac{1}{2}k)$, and its axes parallel to the axes of the ellipse. Since (h, k) is on the ellipse,

$$b^2x^2+a^2y^2-a^2b^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

therefore, adding and subtracting (1) and (2), we find

$$2b^2x^2-b^2hx+a^2ky-a^2b^2 = 0, \quad 2a^2y^2+b^2hx-a^2ky-a^2b^2 = 0 \ldots. (3, 4);$$

showing that A, B, C, D lie also on the parabolas (3) and (4).

Dividing (3) by $2a^2$, (4) by $2b^2$, and adding, we get

$$x^2+y^2-\left(\frac{a^2-b^2}{a^2}\right)\frac{hx}{2}+\left(\frac{a^2-b^2}{b^2}\right)\frac{ky}{2}-\tfrac{1}{2}(a^2+b^2) = 0 \ldots\ldots\ldots (5);$$

or $$\left\{x-\left(\frac{a^2-b^2}{4a^2}\right)h\right\}^2+\left\{y+\left(\frac{a^2-b^2}{4b^2}\right)k\right\}^2 = \tfrac{1}{2}(a+b^2)+\left(\frac{a^2-b^2}{4}\right)^2\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)$$

which shows that the points A, B, C, D lie on a circle, the coordinates of whose centre are $x = \left(\frac{a^2-b^2}{4a^2}\right)h$, $y = -\left(\frac{a^2-b^2}{4b^2}\right)k$,

$$\text{and } (\text{radius})^2 = \tfrac{1}{2}(a^2+b^2)+\left(\frac{a^2-b^2}{4}\right)^2\left(\frac{h^2}{a^4}+\frac{k^4}{b^4}\right).$$

The position of the centre is readily found by drawing the normal at A, cutting the axes in G, g, respectively; then the coordinates of the centre are respectively $\frac{1}{4}$OG and $\frac{1}{4}$Og.

The centre lies on the ellipse $\dfrac{x^2}{\left(\dfrac{a^2-b^2}{4a}\right)^2}+\dfrac{y^2}{\left(\dfrac{a^2-b^2}{4b}\right)^2} = 1.$

The value of r^2 may be put in the form

$$\tfrac{1}{2}(a^2+b^2)+\left(\frac{a^2-b^2}{4}\right)^2\left\{\frac{1}{b^2}-\frac{h^2}{a^2}\left(\frac{a^2-h^2}{a^2b^2}\right)\right\},$$

showing that r^2 is a maximum or minimum according as h is a minimum or maximum; hence,

for a maximum $h = 0$, $k = b$, and therefore $r = \dfrac{a^2+3b^2}{4b}$;

for a minimum $h = a$, $k = 0$, and therefore $r = \dfrac{3a^2+b^2}{4a}$.

If d be the distance between the centre of this circle, and the centre of the circle osculating the conic at A,

$$d^2 = (a^2-b^2)^2\left\{\frac{h^2}{a^4}\left(\frac{h^2}{a^4}-\frac{1}{4}\right)^2+\frac{k^2}{b^4}\left(\frac{k^2}{b^4}-\frac{1}{4}\right)^2\right\},$$

and, as before,

this is a maximum when $h = 0$, $k = b$, and is then $= \frac{9}{16}\left(\dfrac{a^2-b^2}{b}\right)^2$,

and a minimum when $h = a$, $k = 0$, and is then $= \frac{9}{16}\left(\dfrac{a^2-b^2}{a}\right)^2$.

Differentiating the equation to the circle with respect to the variable parameters h, k, we get $\qquad \dfrac{xdh}{a^2} - \dfrac{ydk}{b^2} = 0,$

and by the equation to the ellipse, we get $\dfrac{hdh}{a^2} + \dfrac{kdk}{b^2} = 0$;

therefore $kx + hy = 0$; substituting in the equation to the ellipse

$$h = \pm \frac{abx}{(b^2x^2 + a^2x^2)^{\frac{1}{2}}}, \quad k = \mp \frac{aby}{(b^2x^2 + a^2y^2)^{\frac{1}{2}}};$$

substituting these values in the equation to the circle, and reducing, we find the equation to the envelope to be

$$\left\{x^2 + y^2 - \tfrac{1}{2}(a^2 + b^2)\right\}^2 = \left(\frac{a^2 - b^2}{2}\right)^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right),$$

which represents a bi-circular quartic, having double contact with the ellipse at the extremities of the major and minor axes.

The axes of the parabolas (3) and (4) are given by the equations $x = \frac{1}{2}h$ and $y = \frac{1}{2}k$ respectively, and the intersection of these axes evidently lies on the ellipse $\dfrac{x^2}{(\frac{1}{2}a)^2} + \dfrac{y^2}{(\frac{1}{2}b)^2} = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} = 1$, which is similar to, and concentric with, the given ellipse, and of half the linear dimensions.

The envelope of (3) is the quartic $\left(\dfrac{2x^2 - a^2}{a^2}\right)^2 - \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) = 0$, and the envelope of (4) is $\left(\dfrac{2y^2 - b^2}{b^2}\right)^2 - \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) = 0$; which have double contact with the ellipse at the points where it is cut by the lines

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0.$$

2852. (Proposed by J. J. Walker, M.A.)—The four points being as in 2803, prove that (1) If a circle be described concentric with the conic, and bisecting the intervals on the axis between the foci, the locus of the centre of the circle ABCD is the conic polar-reciprocal to the given one with respect to the above circle. (2.) That the circle ABCD meets the chord common to the conic, and the osculating circle at A, in points equidistant from the centre of the conic. (3.) That the normal at B, and the line joining the centre of the osculating circle at B with the centre of the circle ABCD, are equally inclined to the axis of the conic.

Solution by James Dale.

1. The polar reciprocal of the conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, with respect to the circle $x^2 + y^2 = \frac{1}{2}(a^2 - b^2)$, is the conic $\dfrac{x^2}{\left(\dfrac{a^2 - b^2}{4a}\right)^2} + \dfrac{y^2}{\left(\dfrac{a^2 - b^2}{4b}\right)^2} = 1$, which, as has been already shown, is the locus of the centre of ABCD.

2. The equation to the line joining the centre of the conic to the centre of ABCD is $a^2kx+b^2hy=0$; and the equation to the chord common to the conic and the osculating circle at A is $(y-k)\,a^2k-(x-h)b^2h=0$; and these lines being at right angles, it follows that the common chord cuts ABCD in points equidistant from the centre of the conic.

3. AB being the chord common to ABCD and the osculating circle at B, the line joining the centres of these circles will be at right angles to AB, and the normal at B is at right angles to the tangent at B; but AB and the tangent at B are equally inclined to the axis of the conic; therefore, the lines at right angles to them will also be equally inclined to the axis.

2549. (Proposed by J. Griffiths, M.A.) — Solve the simultaneous equations $\quad yz+x=14,\quad zx+y=11,\quad xy+z=10.$

Solution by J. J. Walker, M.A.

$\left.\begin{array}{r} yz+x=14 \\ zx+y=11 \end{array}\right\}$. Multiply the second by z, and subtract, then $x=\dfrac{11z-14}{z^2-1}$.

Similarly, multiplying the first by z, and subtracting, we have

$$y=\frac{14z-11}{z^2-1},\quad \text{and}\quad xy=\frac{(11z-14)\,(14z-11)}{(z^2-1)^2}.$$

Substituting for xy in the third of the given equations, and reducing, there results $\quad z^5-10z^4-2z^3+174z^2-316z+144=0 \;\ldots\ldots\ldots\ldots(1).$

Again, adding the first two of the given equations, we obtain

$$x+y=\frac{25}{z+1},\quad \text{and from the last } xy=10-z.$$

From these two equations we find

$$2x=\frac{25\pm\sqrt{\left\{625+4\,(z-10)\,(z+1)^2\right\}}}{z+1}=\frac{22z-28}{z^2-1},$$

whence $$\pm\sqrt{\left\{625+4\,(z-10)\,(z+1)^2\right\}}=-3\,\frac{z+1}{z-1}.$$

Squaring both sides, and reducing, there results the same equation (1) as above. Trying the factors of 144, we find that neither 2 nor 3 satisfy (1), but 4 does. Hence

$$z=4,\quad y=\frac{14z-11}{z^2-1}=3,\quad x=\frac{11z-14}{z^2-1}=2$$

are three simultaneous values of z, y, x. The biquadratic which gives the four other values of z is $z^4-6z^3-26z^2+70z-36=0$, which has two imaginary roots, and two real roots, lying between 1 and 2, 8 and 9, respectively.

2867. (Proposed by S. ROBERTS, M.A.)—Being given four circles, show (1) that their twelve centres of similitude are the points of contact of tangents to a cubic curve drawn from three collinear points upon it; (2) that the curve passes also through the intersections of the straight lines joining the centres of the circles two and two together; and (3) that these results constitute a generalization of the theorem, that, four points being given, the intersections of the straight lines joining them two and two, and the bisecting points of those lines limited by the given points, lie upon the same conic.

Solution by the PROPOSER.

The ordinary geometrical construction for the centres of similitude shows that they lie by threes on sixteen lines, and therefore fulfil the conditions of the points of contact of tangents drawn to a cubic curve from collinear points upon it. Moreover, it will be immediately seen, on constructing the figure, that corresponding external and internal centres of similitude are points, the tangents at which meet on the curve.

If we take any triangle as that of reference, the four corners of a quadrilateral, the sides of which intersect at the vertices of the triangle, are determined by the coordinates (α, β, γ), $(\alpha, \beta, -\gamma)$, $(\alpha, -\beta, \gamma)$, $(-\alpha, \beta, \gamma)$. These do not, however, represent the actual values of the perpendiculars from the four points to the lines of reference. Let k_1, k_2, k_3, k_4 be the multipliers necessary to make the coordinates represent the perpendiculars. They can readily be determined, but it is unnecessary to do this for our purpose. Let r_1, r_2, r_3, r_4 be the radii of the circles drawn about the four points as centres. The coordinates of six centres of similitude (internal) may then be written in the forms

$$\left(\frac{r_2}{k_2}+\frac{r_1}{k_1}\right)\alpha,\quad \left(\frac{r_2}{k_2}+\frac{r_1}{k_1}\right)\beta,\quad \left(\frac{r_2}{k_2}-\frac{r_1}{k_1}\right)\gamma;$$
$$\left(\frac{r_3}{k_3}+\frac{r_1}{k_1}\right)\alpha,\quad \left(\frac{r_3}{k_3}-\frac{r_1}{k_1}\right)\beta,\quad \left(\frac{r_3}{k_3}+\frac{r_1}{k_1}\right)\gamma;$$
$$\left(\frac{r_4}{k_4}-\frac{r_1}{k_1}\right)\alpha,\quad \left(\frac{r_4}{k_4}+\frac{r_1}{k_1}\right)\beta,\quad \left(\frac{r_4}{k_4}+\frac{r_1}{k_4}\right)\gamma;$$
$$\left(\frac{r_3}{k_3}+\frac{r_2}{k_2}\right)\alpha,\quad \left(\frac{r_3}{k_3}-\frac{r_2}{k_2}\right)\beta,\quad \left(\frac{r_2}{k_2}-\frac{r_3}{k_3}\right)\gamma;$$
$$\left(\frac{r_4}{k_3}-\frac{r_2}{k_2}\right)\alpha,\quad \left(\frac{r_4}{k_4}+\frac{r_2}{k_2}\right)\beta,\quad \left(\frac{r_2}{k_2}-\frac{r_4}{k_4}\right)\gamma;$$
$$\left(\frac{r_4}{k_4}-\frac{r_3}{k_3}\right)\alpha,\quad \left(\frac{r_3}{k_3}-\frac{r_4}{k_4}\right)\beta,\quad \left(\frac{r_4}{k_4}+\frac{r_3}{k_3}\right)\gamma;$$

and for the six other centres we have the same values α, β, γ, with reciprocal or inverse coefficients.

If now we suppose the equation of the curve to be formed with $\frac{x}{\alpha}, \frac{y}{\beta}, \frac{z}{\gamma}$ in place of x, y, z, we may write X, Y, Z for these quantities, and take away from the above system α, β, γ. The equation will then be such that the points X', Y', Z', $\frac{1}{X'}, \frac{1}{Y'}, \frac{1}{Z'}$ will be "corresponding points," the tangents thereat meeting on the curve.

Take then the equation

$$aX(Y^2+Z^2)+bY(X^2+Z^2)+cZ(X^2+Y^2)+lXYZ=0,$$

and determine it to be satisfied by the coordinates $(1, 1, K_3)$, $(1, K_2, 1)$, $(K_1, 1, 1)$, K_3 being written for $\left(\frac{r_2}{k_2}-\frac{r_1}{k_1}\right)\div\left(\frac{r_2}{k_2}+\frac{r_1}{k_1}\right)$, &c. It will be sufficient to show that the equation will then be also satisfied by another set, say,

$$\frac{r_3}{k_3}+\frac{r_2}{k_2},\ \frac{r_3}{k_3}-\frac{r_2}{k_3},\ \frac{r_2}{k_3}-\frac{r_3}{k_3},\ \text{or}\ \frac{K_2-K_3}{1-K_2K_3},\ 1,\ -1.$$

The rest will follow by symmetry. We must have, then,

$$\begin{vmatrix} \frac{2K_2-K_3}{1-K_2K_3}, & 1+\left(\frac{K_2-K_3}{1-K_2K_3}\right)^2, & -1-\left(\frac{K_2-K_3}{1-K_2K_3}\right)^2, & -\frac{K_2-K_3}{1-K_2K_3} \\ 1+K_3^2\ , & 1+K_3^2\ , & 2K_3\ , & K_3 \\ 1+K_2^2\ , & 2K_2\ , & 1+K_2^2\ , & K_2 \\ 2K_1\ , & 1+K_1^2\ , & 1+K_1^2\ , & K_1 \end{vmatrix}=0,$$

or

$$\begin{vmatrix} \left(1-\frac{K_2-K_3}{1-K_2K_3}\right)^2, & \left(1-\frac{K_2-K_3}{1-K_2K_3}\right)^2, & 0\ , & -\frac{K_2-K_3}{1-K_2K_3} \\ 0\ , & 0\ , & (1+K_3)^2, & K_3 \\ -(1-K_2)^2\ , & -(1-K_2)^2\ , & (1+K_2)^2, & K_2 \\ (1-K_1)^2\ , & -(1-K_1)^2\ , & 4K_1\ , & K_1 \end{vmatrix}=0,$$

or

$$\begin{vmatrix} (1-K_2)^2(1+K_3)^2, & (1-K_2)^2(1+K_3)^2, & 0, & K_3(1+K_2)^2-K_2(1+K_3)^2 \\ 0, & 0, & (1+K_3)^2, & K_3 \\ -(1-K_2)^2, & -(1-K_2)^2, & (1+K_2)^2, & K_2 \\ -(1-K_1)^2, & -(1-K_1)^2, & 4K_1\ , & K_1 \end{vmatrix}=0;$$

and this is identically true, since the first row can be derived from the second and third rows.

The equation of the curve then is of the form, writing L, M, N, P for $X(Y^2+Z^2)$, $Y(X^2+Z^2)$, &c.,

$$\begin{vmatrix} L\ , & M\ , & N\ , & P \\ 1+K_3^2, & 1+K_3^2, & 2K_3\ , & K_3 \\ 1+K_2^2, & 2K_2\ , & 1+K_2^2, & K_2 \\ 2K_1\ , & 1+K_1^2, & 1+K_1^2, & K_1 \end{vmatrix}=0;$$

or (putting for K_1, K_2, K_3 their values, and reducing),

$$\frac{r_1^2}{k_1^2}(L+M+N+2P)+\frac{r_2^2}{k_2^2}(-L-M+N+2P)$$

$$+\frac{r_3^2}{k_3^2}(-L+M-N+2P)+\frac{r_4^2}{k_4^2}(L-M-N+2P)=0.$$

If the circles are equal, six external centres of similitude lie on the line at infinity and the internal centres of similitude, that is to say, the bisect-

ing points of the sides and diagonals of the quadrilateral formed by the four centres of the circles lie on a conic passing through the vertices of the triangle of reference. If these centres are the centres of the inscribed and escribed circles, the equation is its own simple inverse.

2768. (Proposed by J. J. Walker, M.A.)—Let $v^3 - av^2 + bv - c = 0$ be one of the five cubics whose roots are simultaneous values of x, y, z, satisfying the three equations in Question 2549 [viz., $yz + x = 14$, $zx + y = 11$, $xy + z = 10$; see *Reprint*, Vol. IX., p. 66, and Vol. XII., p. 40]; show that b must be one of the five roots of the quintic

$$\beta^5 - 140\beta^4 + 7608\beta^3 - 202044\beta^2 + 2634224\beta - 13531024 = 0,$$

and that the corresponding values of a, c are given by the equations $a + b = 35$, $(32 - b)\,c + (36 - b)\,b = 404$; also that the five values of c are the roots of the quintic

$$\gamma^5 + 4\gamma^4 - 2368\gamma^3 + 61452\gamma^2 - 572256\gamma + 1783296 = 0.$$

Solution by the Proposer.

For, adding the three given equations, $b + a = 35$ ....(1);

again multiplying the same equations, and lastly adding their squares, we find, after substituting for the symmetric functions of x, y, z which enter their values in terms of a, b, c,

$$(b-c)^2 + (a-1)^2 c = 1540 \ldots (2), \quad \text{and} \quad a^2 - 2b + b^2 + 2\,(3-a)\,c = 417 \ldots (3).$$

From (1) and (3),

$$= \frac{404 + b\,(b-36)}{32-b} \;[\text{or } (32-b)\,c + (36-b)\,b = 404] \;\ldots\ldots\ldots (4);$$

and $$c - b = \frac{404 + 2b\,(b-34)}{32-b} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Eliminating a and c from (1), (2), (4), (5),

$$\left\{\frac{404 + 2b\,(b-34)}{32-b}\right\}^2 + \frac{(b-34)^2\left\{404 + b\,(b-36)\right\}}{32-b} = 1540,$$

or $(2b^2 - 68b + 404)^2 + (b-34)^2\,(b^2 - 36b + 404)\,(32-b) - 1540\,(b-32)^2$,

or $$b^5 - 140b^4 + 7608b^3 - 202044b^2 + 2634224b - 13531024 = 0 \;\ldots\ldots (6).$$

This equation is satisfied by $b = 26$, whence and from (1) $a = 9$, and, from (4), $c = 24$. The cubic in v for these values is therefore $v^3 - 9v^2 + 26v - 24 = 0$, the roots of which are 2, 3, 4; and these are values of x, y, z which simultaneously satisfy the three equations in Question 2549.

Again, eliminating a between (1) and (2), developing and arranging, there results $\quad (1 + c)\,b^2 - 70cb + c^2 + 1156c - 1540 = 0,$

and from (4), $\quad b^2 - (36 - c)\,b - 32c + 404 = 0.$

Eliminating b between these two equations, the result is

$$0 = \left\{(c+1)(32c-404)+c^3+1156c-1540\right\}^2$$
$$+\left\{(c+1)(36-c)-70c\right\}\left\{(36-c)(c^3+1156c-1540)+70c(32c-404)\right\},$$

which, developed and arranged, gives

$$c^5+4c^4-2368c^3+61452c^2-572256c+1783296 = 0.$$

This equation will be found, on trial, to be satisfied by $c = 24$.

2843. (Proposed by R. TUCKER, M.A.)—Two parabolas are described through the points A, B, C, D of Question 2803; show that their axes intersect on a similar concentric conic, and that their envelopes and the loci of their vertices and foci are quartic curves.

I. *Solution by* J. J. WALKER, M.A.

Eliminating x^2 between the equations to the conic and circle,

$$2y^2+b^2\left(\frac{x'x}{a^2}-\frac{y'y}{b^2}\right)-b^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is the equation to one of the parabolas through A, B, C, D; the other being

$$2x^2-a^2\left(\frac{x'x}{a^2}-\frac{y'y}{b^2}\right)-a^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Equation (1) may be thrown into the form

$$\left(y-\frac{y'}{4}\right)^2+\frac{b^2x'}{2a^2}(x-\alpha) = 0, \quad \text{where} \quad \alpha = \frac{a^2}{b^2}\cdot\frac{y'^2+8b^2}{8x'} \quad\ldots. (3, 4);$$

and (2) into the form

$$\left(x-\frac{x'}{4}\right)^2+\frac{a^2y'}{2b^2}(y-\beta) = 0, \quad \text{where} \quad \beta = \frac{b^2}{a^2}\cdot\frac{x'^2+8a^2}{8y'} \quad\ldots. (5, 6).$$

From (3), (5) it appears that the equations to the axes of the two parabolas are $4y = y'$, $4x = x'$; dividing these by b, a respectively, squaring and adding, the locus of the intersection is

$$16\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = 1.$$

Again, the coordinates of the vertices are

$$\xi = \frac{a^2}{b^2}\cdot\frac{y'^2+8b^2}{8x'}, \quad \eta = \frac{y'}{4}; \qquad \text{whence } \frac{x'}{a} = \frac{a}{b^2}\cdot\frac{2\eta^2+b^2}{\xi}, \quad \frac{y'}{b} = \frac{4\eta}{b};$$

$$\text{and } \xi' = \frac{x'}{4}, \quad \eta' = \frac{b^2}{a^2}\cdot\frac{x'^2+8a^2}{8y'}; \qquad \text{whence } \frac{x'}{a} = \frac{4\xi'}{a}, \quad \frac{y'}{b} = \frac{b}{a^2}\cdot\frac{2\xi'^2+a^2}{\eta'}.$$

Eliminating x', y' by squaring and adding, the loci of the vertices are

$$a^2(2\eta^2+b^2)^2+b^2\xi^2(16\eta^2-b^2) = 0 \quad \text{and} \quad b^2(2\xi'^2+a^2)^2+a^2\eta'^2(16\xi'^2-a^2) = 0.$$

Lastly, the coordinates of the foci are

$$\mu = \frac{b^4x'^2 + a^4y'^2 + 8a^4b^2}{8a^2b^2x'}, \quad \nu = \frac{y'}{4}, \quad \text{and} \quad \mu' = \frac{x'}{4}, \quad \nu' = \frac{b^4x'^2 + a^4y'^2 + 8a^2b^4}{8a^2b^2y'},$$

whence for the loci of the foci, after eliminating x', y', we have

$$\left\{16\,(a^2-b^2)\,\nu^2 + (8a^2+b^2)\,b^2\right\}^2 + 16a^2b^2\mu^2\nu^2 - a^2b^4\mu^2 = 0,$$

and $$\left\{16\,(b^2-a^2)\,\mu'^2 + (a^2+8b^2)\,a^2\right\}^2 + 16a^2b^2\mu'^2\nu'^2 - a^4b^2\nu'^2 = 0.$$

II. *Solution by the* Proposer.

If δ be the eccentric angle for A, then (Salmon's *Conics*, p. 215) B, C, D may be denoted respectively by $-\frac{1}{3}\delta$, $-\frac{1}{3}\delta + 120°$, $-\frac{1}{3}\delta + 240°$; and the equation to any conic through A, B, C, D will be

$$\left(\frac{x}{a}\cos\tfrac{1}{3}\delta + \frac{y}{b}\sin\tfrac{1}{3}\delta - \cos\tfrac{2}{3}\delta\right)\left(\frac{y}{b}\sin\tfrac{1}{3}\delta - \frac{x}{a}\cos\tfrac{1}{3}\delta - \frac{1}{2}\right)$$

$$= k\left\{\frac{x}{a}\cos\left(60° + \tfrac{1}{3}\delta\right) + \frac{y}{b}\sin\left(60° + \tfrac{1}{3}\delta\right) - \cos\left(60° - \tfrac{2}{3}\delta\right)\right\}$$

$$\times \left\{\frac{y}{b}\sin\left(60° + \tfrac{1}{3}\delta\right) - \frac{x}{a}\cos\left(60° + \tfrac{1}{3}\delta\right) + \frac{1}{2}\right\}.$$

That the curve may represent a parabola gives

$$\left\{k\cos\left(60° + \tfrac{1}{3}\delta\right) - \cos^2\tfrac{1}{3}\delta\right\}\left\{\sin^2\tfrac{1}{3}\delta - \mathrm{K}\sin^2\left(60° + \tfrac{1}{3}\delta\right)\right\} = 0.$$

Taking the former value, we get

$$\frac{y^2}{b^2} + \frac{x}{2a}\cos\delta - \frac{y}{2b}\sin\delta - \frac{1}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

the other parabola is at once seen to be

$$\frac{x^2}{a^2} - \frac{x}{2a}\cos\delta + \frac{y}{2b}\sin\delta - \frac{1}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The respective envelopes are readily seen to be

$$\left(\frac{y^2}{b^2} - \frac{1}{2}\right)^2 = \frac{x^2}{4a^2} + \frac{y^2}{4b^2}, \qquad \left(\frac{x^2}{a^2} - \frac{1}{2}\right)^2 = \frac{x^2}{4a^2} + \frac{y^2}{4b^2}.$$

From (1) and (2) we find the equations to the axes to be

$$4y = b\sin\delta, \quad 4x = a\cos\delta;$$

hence, squaring and adding, we get for locus of intersection

$$16\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 1,$$

a conic similar to and concentric with the given one.

The coordinates of the vertex of (1) are

$$x = a\sec\delta\left(1 + \tfrac{1}{8}\sin^2\delta\right), \quad y = \tfrac{1}{4}b\sin\delta;$$

hence locus of vertex is $\dfrac{x^2}{a^2}\left(1 - \dfrac{16y^2}{b^2}\right) = \left(1 + \dfrac{2y^2}{b^2}\right)^2.$

The abscissa of the focus is $x = \dfrac{8a^2 + a^2\sin^2\delta - b^2\cos^2\delta}{8a\cos\delta}$, whence the locus of the focus of (1) is $64a^2b^2x^2(b^2-16y^2) = [b^2(8a^2-b^2)+16(a^2+b^2)y^2]^2$.

Similar expressions may be obtained for the corresponding loci of (2).

2900. (Proposed by H. M. TAYLOR, B.A.)—The foci of the conic

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

are given by the intersections of two rectangular hyperbolas concentric with the original conic, whose equations are

$$(b^2-ac)(x^2-y^2) + 2(be-cd)x + 2(ae-bd)y + (a-c)f - (d^2-e^2) = 0,$$

and $$(b^2-ac)xy + (bd-ae)x + (be-cd)y + bf - de = 0.$$

I. *Solution by* JAMES DALE.

For x and y put $x'+h$, $y'+k$, and the equation to the conic becomes

$$ax'^2 + 2bx'y' + cy'^2 + 2(ah+bk+d)x' + 2(bh+ck+e)y'$$
$$+ ah^2 + 2bhk + ck^2 + 2dh + 2ek + f = 0;$$

and if (h, k) be a focus, and as usual $i = \sqrt{(-1)}$, then $y - ix = 0$ will be a tangent; therefore the equation

$$(a+2ib-c)x^2 + 2\left\{(ah+bk+d) + i(bh+ck+e)\right\}x$$
$$+ ah^2 + 2bkh + ck^2 + 2dh + 2ek + f = 0$$

will have equal roots; therefore

$$(ah+bk+d)^2 + 2i(ah+bk+d)(bh+ck+e) + (bh+ck+e)^2$$
$$= (a+2ib-c)(ah^2+2bhk+ck^2+2dh+2ek+f);$$

and equating the possible and impossible parts of this equation, we get

$$(b^2-ac)(h^2-k^2) + 2(be-cd)h + 2(ae-bd)k + (a-c)f - (d+e)(d-e) = 0 \quad \text{............(1)},$$

and $$(b^2-ac)(hk) + (bd-ae)h + (be-cd)k + (bf-de) = 0 \quad \text{............(2)}.$$

Equation (1) represents a rectangular hyperbola concentric and coaxal with the given conic; (2) represents a rectangular hyperbola concentric with and having its asymptotes coincident with the axes of the given conic.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

If (X, Y) be a focus of a conic, the rectangle under the perpendiculars from (X, Y) on any two parallel tangents is constant; or if these tangents be $lx+my = p_1$, and $lx+my = p_2$, we must have

$$\frac{(lX+mY-p_1)(lX+mY-p_2)}{l^2+m^2} = \mu \quad \text{........................(A)}.$$

Now the condition that $lx+my=p$ may touch the conic $ax^2+2bxy+cy^2+2dx+2ey+f=0$ is the condition that

$$ax^2+2bxy+cy^2+2(dx+ey)\frac{lx+my}{p}+f\left(\frac{lx+my}{p}\right)^2=0$$

may give equal roots in $x:y$,

or $$\left(a+\frac{2dl}{p}+\frac{fl^2}{p^2}\right)\left(c+\frac{2lm}{p}+\frac{fm^2}{p^2}\right)=\left(b+\frac{dm+el}{p}+\frac{flm}{p^2}\right)^2;$$

or $$0=(ac-b^2)p^2+2p\left\{l(cd-be)+m(ae-bd)\right\}+l^2(cf-e^2)$$
$$+2lm(ed-bf)+m^2af-d^2),$$

an equation whose roots are p_1, p_2; and substituting in (A), we must have the equation

$$\left\{(lX+mY)^2-\mu(l^2+m^2)\right\}(ac-b^2)+2(lX+mY)\left\{l(cd-be)+m(ae-bd)\right\}$$
$$+l^2(cf-e^2)+2lm(ed-bf)+m^2(af-d^2)=0$$

true for all values of $l:m$. Whence

$$(X^2-\mu)(ac-b^2)+2X(cd-be)+cf-e^2=0,$$
$$(Y^2-\mu)(ac-b^2)+2Y(ae-bd)+af-d^2=0,$$
$$XY(ac-b^2)+X(ae-bd)+Y(cd-be)+ed-bf=0.$$

It is obvious that all conics through the four foci of a given conic are rectangular hyperbolas concentric with the given conic; because the four are at the corners of a parallelogram, and each is the centre of perpendiculars of the triangle formed by the other three. [See Salmon's *Conics*, 5th ed., Art. 258, and Wolstenholme's *Book of Mathematical Problems*, Question 690.]

2820. (Proposed by R. Tucker, M.A.)—Find the locus of the intersection of perpendicular normals of a Lemniscate.

Solution by the Proposer.

Taking for equations to the Lemniscate

$$x=a\cos\theta\cos^{\frac{1}{2}}2\theta,\quad y=a\sin\theta\cos^{\frac{1}{2}}2\theta,$$

we have for the normals at θ and ϕ

$$y-x\tan 3\theta=-a\sin 2\theta\cos^{\frac{1}{2}}2\theta\sec 3\theta,$$
$$y-x\tan 3\phi=-a\sin 2\phi\cos^{\frac{1}{2}}2\phi\sec 3\phi,$$

for perpendicularity $\phi=\frac{1}{6}\pi+\theta$;

and the equations become

$$y\cos 3\theta-x\sin 3\theta=-a\sin 2\theta\cos^{\frac{1}{2}}2\theta\ \ldots\ldots\ldots\ldots(1),$$
$$y\sin 3\theta+x\cos 3\theta=a\sin(\tfrac{1}{3}\pi+2\theta)\cos^{\frac{1}{2}}(\tfrac{1}{3}\pi+2\theta)\ \ldots\ldots\ldots\ldots(2).$$

Squaring and adding, we have

$$x^2+y^2=\frac{\sqrt{3}}{4}a^2\cos\left(\tfrac{1}{3}\pi+2\theta\right)=\frac{\sqrt{3}}{4}\lambda a^2\ \ldots\ldots\ldots\ldots\ldots\ldots(a);$$

from (a) $\quad +\sin 6\theta=3\lambda-4\lambda^3,\quad \cos 6\theta=(1-4\lambda^2)\sqrt{(1-\lambda^2)},$

$$\cos 2\theta=\frac{\sqrt{3}\,.\,\lambda\pm\sqrt{(1-\lambda^2)}}{2},\quad \sin 2\theta=\frac{\pm\sqrt{3}\sqrt{(1-\lambda^2)}-\lambda}{2}.$$

Squaring (1), we have

$$x^2+y^2+(y^2-x^2)\cos 6\theta-2xy\sin 6\theta=2a^2\sin^2 2\theta\cos 2\theta,$$

or $\quad x^2+y^2+(y^2-x^2)(1-4\lambda^2)\sqrt{(1-\lambda^2)}-2xy(3\lambda-4\lambda^3)$

$$=a^2\left(\sqrt{3}\,.\,\lambda\pm\sqrt{1-\lambda^2}\right)\frac{3-2\lambda^2\mp2\sqrt{3}\,.\,\lambda\sqrt{(1-\lambda^2)}}{4}\ \ldots\ldots(\beta),$$

in which equation, if we put for λ its value, we have the solution required.

The curve may be traced by its polar equations

$(1)\ \ldots\ldots\ldots\ \rho\sin(3\theta-\phi)=a\sin 2\theta\sqrt{(\cos 2\theta)},$

$(a)\ \ldots\ldots\ldots\ \rho^2=\frac{\sqrt{3}}{4}a^2\cos\left(\tfrac{1}{3}\pi+2\theta\right);$

with this substitution, (β) becomes

$$\left[\rho^2\cos 2\phi\left(1-\frac{64\rho^4}{3a^4}\right)\mp\left(\frac{32\rho^4}{3a^2}-\frac{3a^2}{4}\right)\right]\sqrt{\left(1-\frac{16\rho^4}{3a^4}\right)}$$
$$-\rho^2\sin 2\phi\,\frac{4\rho^2}{\sqrt{3}\,.\,a^2}\left(3-\frac{64\rho^4}{3a^2}\right)=0.$$

2930. (Proposed by Professor SYLVESTER.)—Given

$$x+y+z-\xi-\eta-\zeta=i+j+k=i'+j'+k',$$
$$xy+xz+yz-\xi\eta-\xi\zeta-\eta\zeta=i(y+z)+j(z+x)+k(x+y)$$
$$=i'(\eta+\zeta)+j'(\zeta+\xi)+k'(\xi+\eta),$$
$$xyz-\xi\eta\zeta=iyz+jzx-kxy=i'\eta\zeta+j\,\zeta\xi+k'\xi\eta;$$

prove that
$$\frac{x-y)^2(x-z)^2(y-z)^2}{(\xi-\eta)^2(\xi-\zeta)^2(\eta-\zeta)^2}=\frac{i'j'k'}{ijk},$$

and that, in like manner, if there be any number of quantities $x, y, z, t\ldots,$ and an equal number $\xi, \eta, \zeta, \tau\ldots,$ connected together in a manner similar to the above, then

$$\frac{(x-y)^2(x-z)^2(x-t)^2(y-z)^2(y-t)^2(z-t)^2}{(\xi-\eta)^2(\xi-\zeta)^2(\xi-\tau)^2(\eta-\zeta)^2(\eta-\tau)^2(\zeta-\tau)^2}=\frac{i'j'k'l'\ldots}{ijkl\ldots}.$$

Solution by W. H. LAVERTY, B.A.

Let
$$x+y+z=a_1;\quad yz+zx+xy=b_1;\quad xyz=c_1;$$
$$\xi+\eta+\zeta=\alpha_1;\quad \eta\zeta+\zeta\xi+\xi\eta=\beta_1;\quad \xi\eta\zeta=\gamma_1;$$
$$a_1-\alpha_1=A;\quad b_1-\beta_1=B;\quad c_1-\gamma_1=C.$$

Now $$X^3-a_1X^2+b_1X-c_1=(X-x)(X-y)(X-z);$$
therefore $$x^3-a_1x^2+b_1x-c_1=0,$$
and $$x^3-a_1x^2+\beta_1x-\gamma_1=(x-\xi)(x-\eta)(x-\zeta).$$

Subtracting, $$Ax^2-Bx+C=(x-\xi)(x-\eta)(x-\zeta);$$
similarly, $$A\xi^2-B\xi+C=(\xi-x)(\xi-y)(\xi-z);$$
therefore $$\frac{(Ax^2-Bx+C)(Ay^2-By+C)(Az^2-Bz+C)}{(A\xi^2-B\xi+C)(A\eta^2-B\eta+C)(A\zeta^2-B\zeta+C)}=1.$$

Now
$$\begin{aligned} i \quad +j \quad +k \quad &= A,\\ i(y+z)+j(z+x)+k(x+y) &= B,\\ i.yz \quad +j.zx \quad +k.xy \quad &= C; \end{aligned}$$
and multiplying the first of these by X^2, and the second by X, and subtracting the second from the sum of the first and third, the coefficient of i is $(X-y)(X-z)$; and similarly for the coefficients of j and k.

Therefore
$$\begin{aligned} Ax^2-Bx+C &= i(x-y)(x-z),\\ Ay^2-By+C &= j(y-z)(y-x),\\ Az^2-Bz+C &= k(z-x)(z-y); \end{aligned}$$
and similarly for i', j', k'.

$$\therefore \frac{i'.j'.k'}{i.j.k}=\frac{(A\xi^2-\ldots)(A\eta^2-\ldots)(A\zeta^2-\ldots)}{(Ax^2-\ldots)(Ay^2-\ldots)(Az^2-\ldots)}\cdot\frac{(y-z)^2(z-x)^2(x-y)^2}{(\eta-\zeta)^2(\zeta-\xi)^2(\xi-\eta)^2}$$
$$=\frac{(y-z)^2(z-x)^2(x-y)^2}{(\eta-\zeta)^2(\zeta-\xi)^2(\xi-\eta)^2}.$$

The solution will evidently be precisely the same for four or any number of quantities x, y, z, t ....

2783. (Proposed by M. JENKINS, M.A.)—A series of n letters consists of m_1 groups each containing p_1 letters, m_2 groups each containing p_2 letters, &c.; show that the number of combinations of the n letters take r at a time, under the restriction that no two letters of the same group shall enter into the same combination, is

$$\Sigma \frac{\lfloor m_1}{\lfloor\alpha . \lfloor m_1-\alpha}\cdot\frac{\lfloor m_2}{\lfloor\beta \lfloor m_2-\beta}\ldots\ldots p_1^{\alpha}p_2^{\beta}\ldots\ldots,$$

where $\alpha, \beta, \ldots$ &c. take all positive integral values, less than $m_1, m_2, \ldots$ respectively, consistent with the relation $\alpha+\beta+\gamma+\ldots=r$.

Solution by the PROPOSER.

The coefficient of x^r in the continued product of

$$\{1+(a+a'+a''+\ldots)x\}\{1+(b+b'+b''+\ldots)x\}\ldots\ldots \text{\&c.}$$

gives the combinations required if $a, a', a''\ldots$ be one group, $b, b', b''\ldots$ another group, and so on. The number of the combinations is found by

putting $a = a' = a'' = \ldots = 1 = b = b' = b'' \ldots$ &c., and is therefore the coefficient of x^r in $(1+p_1x)^{m_1}(1+p_2x)^{m_2}\ldots$, if m_1 of the groups contain the same number of letters, viz. p_1, and so on; and this coefficient is the above-named series, which proves the theorem.

[The particular case in which $p_1 = 1$, $p_2 = 2$, $p_3 \ldots$ &c. $= 0$, has been proved independently by Prince CAMILLE DE POLIGNAC, in a paper read by him before the London Mathematical Society.]

2389. (Proposed by Professor CREMONA.)—Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère recontrent en quatre points harmoniques toute conique inscrite dans le quadrilatère.

Solution by the Rev. R. TOWNSEND, F.R.S.

Denoting by M and N any pair of lines dividing harmonically the three diagonals of the quadrilateral; by A and A′, B and B′, C and C′ the three pairs of concurrent lines passing through their six extremities; and by T and T′ the pair of concurrent tangents to any conic inserted in the quadrilateral; then, since, by a well-known property, the four pairs of concurrent lines A and A′, B and B′, C and C′, T and T′ are in involution, and since, by hypothesis, M and N divide harmonically the three angles AA′, BB′, CC′, therefore they divide harmonically the angle TT′; and therefore, &c.

2821. (From WOLSTENHOLME'S *Book of Mathematical Problems.*)—Show that the mean value of the distance from one of the foci of all points within a given prolate spheroid is $\frac{1}{4}a(3+e^2)$, $2a$ being the axis and e the eccentricity.

Solution by R. W. GENESE.

The mean value of the focal distances of all points of the surface from the focus of a prolate spheroid is a: for, clearly, if S and H be the foci, we have average of SP = average of HP over same surface

$$= \text{average of } \tfrac{1}{2}(\text{SP}+\text{HP}) = a.$$

Let α, β be the axes of a confocal spheroid, whose volume therefore is $\pi\alpha\beta^2$; and $\alpha^2-\beta^2 =$ constant $= c^2$. The volume of a confocal stratum is therefore

$$\frac{d}{d\alpha}\pi\alpha(\alpha^2-c^2) = \pi(3\alpha^2-c^2)\,d\alpha;$$

hence average value of SP $= \dfrac{\int_c^a \alpha\,.\,\pi(3\alpha^2-c^2)\,d\alpha}{\pi ab^2}$

$$= \frac{\frac{3}{4}(a^4-c^4)-\frac{1}{2}c^2(a^2-c^2)}{ab^2} = \frac{3(a^2+c^2)-2c^2}{4a}$$

$$= \tfrac{1}{4}a(3+e^2), \text{ since } c = ae.$$

[A Solution by Mr. TUCKER is given on p. 87 of Vol. XI. of the *Reprint.*]

2922. (Proposed by M. W. Crofton, F.R.S.)—

If $\quad x_1 = \dfrac{x_0}{1+a_0x_0}, \quad x_2 = \dfrac{x_1}{1+a_1x_1}, \quad x_3 = \dfrac{x_2}{1+a_2x_2}, \quad$ &c.,

prove that $x_n = \dfrac{x_0}{1+hx_0}$, where $h = a_0 + a_1 + \dots + a_{n-1}$.

Solution by the Rev. R. Harley, F.R.S.; A. B. Evans, M.A.; *and others.*

If $\quad x_p = \dfrac{x_{p-1}}{1+a_{p-1}x_{p-1}}$, we have

$$a_0 = \frac{1}{x_1} - \frac{1}{x_0}, \quad a_1 = \frac{1}{x_2} - \frac{1}{x_1}, \quad \dots\dots\dots \quad a_{n-1} = \frac{1}{x_n} - \frac{1}{x_{n-1}};$$

therefore $\qquad \Sigma a = \dfrac{1}{x_n} - \dfrac{1}{x_0}, \quad$ or $\quad x_n = \dfrac{x_0}{1+x_0\Sigma a}$.

So that, whatever a_0, a_1, a_2, &c. may be, the theorem is true.

2615. (Proposed by R. W. Symes, B.A.)—If

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0$$

be an algebraical equation whose roots are $\alpha, \beta, \gamma, \dots$, find the equation whose roots are the different values of

$$\frac{dp_1}{d\alpha} + \frac{dp_2}{d\alpha} + \dots\dots + \frac{dp_n}{d\alpha}.$$

Solution by the Proposer.

Let $\qquad y = \dfrac{dp_1}{d\alpha} + \dfrac{dp_2}{d\alpha} + \dots\dots + \dfrac{dp_n}{d\alpha}.$

Now, in my solution of Question 2367 (*Reprint*, Vol. IX., p. 69), I have shown that

$$-\frac{dp_1}{d\alpha} = 1, \quad -\frac{dp_2}{d\alpha} = \alpha + p_1, \quad -\frac{dp_3}{d\alpha} = \alpha^2 + p_1\alpha + p_2, \ \dots\dots$$

Therefore the required equation is obtained by eliminating x between the given equation and the following:—

$$y + 1 + (x+p_1) + (x^2+p_1x+p_2) + \dots + (x^{n-1}+p_1x^{n-2} + \dots + p_{n-1}) = 0,$$

or $\quad y + (1+x+x^2+\dots+x^{n-1}) + p_1(1+x+x^2+\dots+x^{n-2}) + \dots + p_{n-1} = 0,$

or $\qquad y + \dfrac{1-x^n}{1-x} + p_1 \cdot \dfrac{1-x^{n-1}}{1-x} + p_2 \cdot \dfrac{1-x^{n-2}}{1-x} + \dots + p_{n-1} = 0,$

or $y+\frac{1}{1-x}\left\{(1+p_1+p_2+\ldots+p_n)-(x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n)\right\}=0,$

or $y+\frac{a}{1-x}=0$ (where a = sum of the coefficients of the given equation),

or $x=\frac{a+y}{y}$; consequently the required equation is

$$(y+a)^n+p_1(y+a)^{n-1}y+\ldots+p_{n-1}(y+a)y^{n-1}+p_ny^n=0.$$

2790. (Proposed by A. Martin.)—Find the mean distance of all the points in a right circular cylinder from one end of the axis.

Solution by the Proposer.

Let a be the length of the cylinder, r the radius of its base, (x, y) the coordinates of any point within the cylinder, the axis of x coinciding with the axis of the cylinder, and M the mean distance required. Then

$$M=\frac{\int_0^a\int_0^r 2\pi y\,(x^2+y^2)^{\frac{1}{2}}\,dx\,dy}{\pi ar^2}=\frac{1}{ar^2}\int_0^a\int_0^r y\,(x^2+y^2)^{\frac{1}{2}}\,dx\,dy$$

$$=\frac{2}{3ar^2}+\int_0^a\left\{(x^2+r^2)^{\frac{3}{2}}-x^3\right\}dx$$

$$=\frac{2}{3ar^2}\int_0^a x^2(x^2+r^2)^{\frac{1}{2}}\,dx+\frac{2}{3a}\int_0^a(x^2+r^2)^{\frac{1}{2}}\,dx-\frac{1}{3ar^2}\int_0^a x^3\,dx$$

$$=\frac{(2a^2+5r^2)(a^2+r^2)^{\frac{1}{2}}}{12r^2}-\frac{a^3}{6r^2}+\frac{r^2}{4a}\log\left(\frac{a+(a^2+r^2)^{\frac{1}{2}}}{r}\right).$$

2932. (Proposed by the Rev. J. Wolstenholme, M.A.)—Given the inscribed and circumscribed circles of a triangle, the envelope of the polar circle is a bicircular quartic.

I. *Solution by* W. K. Clifford, B.A.

Let B, C, X be the inscribed, circumscribed, and polar circles respectively. The circle X has to be such that a triangle self-conjugate with regard to it can be circumscribed to B and inscribed to C; that is, it is subject to a condition of the first and a condition of the second degree in its coefficients. Two such circles can therefore be drawn through an arbitrary point.

Now, *any series of circles, such that two of them can be drawn through an arbitrary point, is one system of bitangent circles of a bicircular quartic.* For

the equation of a circle of the series must contain a variable parameter in the second order; that is, it must be of the form

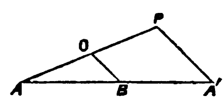
$$*X + 2\theta Y + \theta^2 Z = 0,$$

where $X = 0$, $Y = 0$, $Z = 0$ are circles. But the envelope of this is $XZ = Y^2$, a bicircular quartic.

This important remark is made by CREMONA (*Teoria Geometrica*, ii. 21) in the case of a series of curves of any order and of index 2; that is, such that two of them can be drawn through an arbitrary point. The envelope is always what Prof. CAYLEY (*Edinb. Phil. Trans.* 1868) calls a *trizomal* curve; viz. X, Y, Z being any three curves of the series, its equation may be written

$$\sqrt{\alpha X} + \sqrt{\beta Y} + \sqrt{\gamma Z} = 0.$$

II. *Solution by the* PROPOSER.



Let A, B be the centres of the given circumscribed and inscribed circles, a, b their radii, $AB = \delta \equiv (a^2 - 2ab)^{\frac{1}{2}}$, P the centre of the polar circle of one of the triangles; then if O be middle point of AP, $OB = \frac{1}{2}a - b$, because O is the centre of the nine-point circle of radius $\frac{1}{2}a$ which touches the inscribed circle. Hence, if AB be produced to A′, so that $BA' = AB$, A′ is a fixed point, and $A'P = a - 2b$, or the locus of P is a circle. Now take A′ for origin of polar coordinates, and let ρ be the radius of the polar circle whose centre is P; then $AP^2 = a^2 + 2\rho^2$, whence the equation of the polar circle is, ($\angle PA'A \equiv \alpha$,)

$$\rho^2 = r^2 + (a-2b)^2 - 2r(a-2b)\cos(\theta - \alpha),$$

also

$$a^2 + 2\rho^2 = (a-2b)^2 + 4\delta^2 - 4\delta(a-2b)\cos\alpha;$$

whence the equation of the polar circle, involving only the parameter α, is

$$r^2 - \delta^2 + 2b^2 = 2(a-2b)\left\{r\cos(\theta-\alpha) - \delta\cos\alpha\right\},$$

and for the envelope $\quad 0 = 2(a-2b)\left\{r\sin(\theta-\alpha) + \delta\sin\alpha\right\};$

or the equation of the envelope is

$$(r^2 - \delta^2 + 2b^2)^2 = 4(a-2b)^2(r^2 - 2r\delta\cos\theta + \delta^2);$$

or, in rectangular coordinates, A′ origin and A′A axis of x, is

$$(x^2 + y^2 - \delta^2 + 2b^2)^2 = 4(a-2b)^2\left\{(x-\delta)^2 + y^2\right\},$$

a bicircular quartic, in fact a Cartesian, of which B is one focus.

2623. (Proposed by W. S. M'CAY, B.A.)—If a conic pass through the four points of contact of tangents to a cubic from a point (A) on the curve, and through two other points (B, C) on the cubic; then A is the pole of BC with regard to the conic.

* [Mr. WOLSTENHOLME remarks that "it is not necessary that X, Y, Z should be all circles; it is sufficient that one be a circle and the others straight lines. Thus the envelope might, so far as depends on this reasoning, be a circular cubic."]

I. *Solution by the* Rev. R. Townsend, F.R.S.

This property is evidently the converse of the following:—"If a variable conic pass through four fixed points, the locus of the points of contact of tangents to it from a fifth fixed point is a cubic, passing through the given points, and touching at the four the four lines connecting them with the fifth," which may be readily proved as follows:—

The equation of the variable conic being $U+kV=0$, where U and V are any two fixed conics passing through the four points, and k the variable parameter; if $(\alpha', \beta', \gamma')$ be the coordinates of the fifth point, and Δ the symbol of operation $\alpha' \frac{d}{d\alpha} + \beta' \frac{d}{d\beta} + \gamma' \frac{d}{d\gamma}$; then, since at the points of contact of the two tangents from $(\alpha', \beta', \gamma')$ to $U+kV$ we have $\Delta U + k . \Delta V = 0$; therefore, eliminating k, we have $U . \Delta V - V . \Delta U = 0$, and therefore &c. as regards the locus being a cubic passing through the four points. And since, at the points of contact of the four tangents to it from $\alpha'\beta'\gamma'$, we have $U . \Delta^2 V - V . \Delta^2 U = 0$, therefore &c., as regards its touching the four lines. That it passes through $(\alpha', \beta', \gamma')$ is evident *à priori*, a conic of the variable system passing through it.

N.B.—The corresponding property in Geometry of Three Dimensions, —viz., "If a variable quadric pass through a fixed quartic, the locus of the points of contact of tangents to it from a fixed point is a cubic, passing through the quartic and point, and touching along the former the cone connecting it with the latter,"—may evidently be proved in precisely the same manner.

II. *Solution by the* Proposer.

Let U, V be two conics, P, Q the polars of a fixed point A with regard to them; then the locus of the points of contact of tangents from A to the system of conics through the intersections of U, V is $UQ-PV=0$, a cubic.

But the lines joining A to these intersections touch this cubic; for their equation is $$(UV'-VU')^2 = 4\,(UQ-PV)\,(PV'-QU')$$ (the dotted letters being the result of substituting the coordinates of A), a locus evidently touching the cubic at the four points.

[A Solution by Mr. Thomson is given on p. 68 of Vol. XI. of the *Reprint*.]

2917. (Proposed by the Rev. J. Wolstenholme, M.A.)—If a conic pass through two given points and touch a given conic at a given point, the chord of intersection with the given conic will pass through a fixed point on the given straight line.

I. *Solution by* T. Cotterill, M.A.

More generally, if a, b are two points on a given conic, and c, d two other points not on the conic, then the conic and straight line through c, d form a cubic. Any conic through a, b, c, d cuts the given conic again in two points which are collinear with a fixed point on the line through c, d; viz., the *opposite* of the four points a, b, c, d on the cubic.

II. *Solution by* F. D. Thomson, M.A.; J. Dale; *and others.*

If a circle cut a conic in four points, the pairs of common chords are known to be equally inclined to the major axis of the conic, since the rectangles contained by their segments are equal, and these rectangles are proportional to the parallel diameters.

Hence, when two of the four points coincide, the common chord to the conic and the circle is equally inclined with the common tangent to the major axis.

Therefore the common chord made by a variable circle touching the conic at a given point is parallel to a fixed direction; hence generally, replacing the circular points at infinity by any two fixed points, we see that the common chord in the proposed question passes through a fixed point on the line joining the two given points.

[Mr. Tucker remarks that the theorem is a particular case of the theorem, "Given four points on a conic, its chord of intersection with a fixed conic passing through two of these points will pass through a fixed point," which is readily seen to lie on the fixed line. Both results follow from Salmon's *Conics*, § 266.]

2907. (Proposed by M. W. Crofton, F.R.S.)—Given that three events A, B, C occurred in a certain century, A having preceded B, and B having preceded C, this being all the information obtainable, show that the probability (p) that the date of the event B was not distant more than n years from the middle of the century is

$$p = 3\frac{n}{100} - 4\left(\frac{n}{100}\right)^3.$$

I. *Solution by* Stephen Watson.

The total number of years on which A can fall (n being constant) is $50+n$, and the like number for C is $50-n$; hence the required chance is

$$p = \frac{\int_{-n}^{n}(50^2-n^2)\,dn}{\int_{-50}^{50}(50^2-n^2)\,dn} = \frac{50n-\frac{1}{3}n^3}{50^3-\frac{1}{3}50^3} = \frac{3n}{100} - 4\left(\frac{n}{100}\right)^3.$$

II. *Solution by the* Rev. J. Wolstenholme, M.A.

If x, $x+y$, $x+y+z$ be the dates (in the century) of the three events, then all that is known is that x, y, z are all positive, and $x+y+z < 100$. The whole possible cases may then be represented by the positions of points lying within the tetrahedron formed by the coordinate planes and the plane $x+y+z = 100$; and the favourable cases will correspond to points lying be-

tween the two planes $x+y=50+n$, $x+y=50-n$. The chance will then be

$$\int_{50-n}^{50+n} u(100-u)\,du + \int_{0}^{100} u(100-u)\,du,$$

or
$$\int_{1-m}^{1+m} u(2-u)\,du + \int_{0}^{2} u(2-u)\,du, \quad \text{if } m \equiv \tfrac{1}{50}n;$$

or
$$\tfrac{3}{2}m - \tfrac{1}{2}m^3, \text{ which } = \tfrac{3}{100}n - 4\left(\tfrac{1}{100}n\right)^3.$$

If $\frac{1}{100}n = \sin\alpha$, the required chance is $\sin 3\alpha$.

2950. (Proposed by Professor SYLVESTER.)—If j be any positive integer, then

$$\frac{j+1}{1.j} + \frac{j+1}{2(j-1)} + \frac{j+1}{3(j-2)} + \dots\dots + \frac{j+1}{j.1} = 2\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots\dots + \frac{1}{j}\right).$$

Solution by the Rev. R. HARLEY, F.R.S.

We have
$$\frac{j+1}{1.j} = 1 + \frac{1}{j}, \qquad \frac{j+1}{2(j-1)} = \frac{1}{2} + \frac{1}{j-1},$$
$$\frac{j+1}{3(j-2)} = \frac{1}{3} + \frac{1}{j-2}, \qquad \dots\dots\dots \qquad \frac{j+1}{j.1} = \frac{1}{j} + \frac{1}{1};$$

and, since $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots\dots + \frac{1}{j} = \frac{1}{j} + \frac{1}{j-1} + \frac{1}{j-2} + \dots\dots + \frac{1}{1}$,

the truth of the theorem is manifest.

Mr. TUCKER and others give the proof thus:—

Assume
$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots\dots + \frac{1}{j-1} + \frac{1}{j} = S,$$

then
$$\frac{1}{j} + \frac{1}{j-1} + \frac{1}{j-2} + \dots\dots + \frac{1}{2} + \frac{1}{1} = S;$$

adding, we have

$$\frac{j+1}{1.j} + \frac{j+1}{2(j-1)} + \frac{j+1}{3(j-2)} + \dots\dots + \frac{j+1}{j.1} = 2S = 2\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots\dots + \frac{1}{j}\right).$$

2923. (Proposed by S. ROBERTS, M.A.)—In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle on which four foci lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle.

Solution by W. K. Clifford, B.A.

A bicircular quartic is its own inverse with regard to any focal circle (Moutard, *Nouvelles Annales*, 1866). The bitangent circles divide themselves into four systems, all the circles of any one system being cut orthogonally by the corresponding focal circle. Through any point of the plane can be drawn two bitangent circles of each system. The two bitangent circles, then, that can be drawn through the centre of the focal circle of their system, are in fact straight lines touching the curve in two pairs of inverse points, which consequently lie on a circle.

The corresponding theory in anallagmatic surfaces is that the centre of each one of the five principal spheres is vertex of a quadric cone doubly tangent to the surface; the curve of contact being the intersection of this cone with a sphere. These five cones noticed by Moutard are independently arrived at by Kummer (*Berlin. Monatsber.*) in the case of the general quartic surface with a nodal conic.

2603. (Proposed by T. Cotterill, M.A.)—In a plane, (1) on every line, there are two points harmonic conjugates to the conics through four points A, B, C, D. They are real, if the product of the four triangles ABC, BCD, CDA, DAB, and the perpendiculars from the points A, B, C, D on the line, is positive. If a line does not cut a conic, why are the conjugates on the line real, to any set of four points on the conic? Give a rule for determining the nature of the points on a line, when the product of the triangles is (i) positive, (ii) negative.

(2.) Through any point, there are two lines conjugate to the conics inscribed in four lines. They are real in those regions of the plane formed by the four lines in which the product of perpendiculars from the point has the same sign as that of points in the convex quadrilateral. Hence every point on or within a conic inscribed in the four lines lies in these regions.

Solution by W. H. Laverty, B.A.

1. The line always cuts the conic and sides of the quadrilateral in points in involution, the double points (real or imaginary) of which are harmonic conjugates to the conics. Now if AA_1, BB_1, CC_1, DD_1 be the perpendiculars respectively from A, B, C, D on BC, DA, DA, BC, and AA_2, BB_2, CC_2, DD_2 those on the given line; then the product of the triangles is of the same sign as $AA_1 . BC . DD_1 . BC . CC_1 . DA . BB_1 . DA$, that is, of the same sign as $AA_1 . BB_1 . CC_1 . DD_1$. But

$AA_1 . BB_1 . CC_1 . DD_1$ is positive for the Euclidean skew quadrilateral ... (1),
and negative for the reentrant quadrilateral ......(2);

and according as $AA_2 . BB_2 . CC_2 . DD_2$ is positive or negative,
so in (1) the system is non-overlapping or overlapping,
and in (2) the system is overlapping or non-overlapping.

Consequently the system is non-overlapping or overlapping, *i. e.*, has real or imaginary double points, according as the two products have the same or opposite signs, *i. e.*, according as the given product of triangles and perpendiculars is positive or negative.

If the line does not cut the conic, the two points where the line meets the conic are imaginary, and the double points which are harmonic conjugates of these are real.

A simple analytical proof of this is as follows:—The equations to the conic and line may be put into the forms

$$\beta^2 + C\gamma^2 + 2k\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0, \quad \text{and} \quad \alpha = 0,$$

where k is indeterminate. For the points where α cuts the conic we have

$$\beta^2 + 2k\beta\gamma + C\gamma^2 = 0\,; \quad \text{therefore} \quad \beta = \gamma\left\{-k \pm (k^2 - C)^{\frac{1}{2}}\right\},$$

which can only be imaginary when C is positive. But for the double rays we have $\quad k = \pm C^{\frac{1}{2}}$; therefore $\beta = C^{\frac{1}{2}}\gamma$, $\beta = -C^{\frac{1}{2}}\gamma$,

which are always possible when C is positive, that is, when the line does not meet the conic.

2. The first part of this is the exact correlative of the first part of (1). Lines drawn from a point to the vertices of a quadrilateral form an involution of which the tangents to the conic are a pair; and the double rays are conjugate lines of the conic.

For any point inside the convex quadrilateral, the involution of lines is non-overlapping, and therefore the double lines are real. And it is easily seen that, if we move into those regions where we alter the sign of only one perpendicular, the involution becomes overlapping and the double rays imaginary. Also, for any point inside the conic, the double rays are real, as in (1); and as for a point on the conic, the tangent is a double ray, they are real there also. Therefore any point on or within the conic must lie in the stated regions.

[A Solution by Mr. Wolstenholme is given on p. 28 of this volume of the *Reprint*.]

2399. (Proposed by the Rev. R. Townsend, F.R.S.)—Find a point on a sphere such that the triangle determined by the middle points of the three arcs connecting it with three given points on the sphere will have two of its sides given.

Solution by the Proposer.

If A, B, C be the three given points, O the required point, and X, Y, Z the three middle points of the three arcs OA, OB, OC; then, since, by a well-known property,

$$\cos YZ = \cos \tfrac{1}{2} BC \,.\, \cos \tfrac{1}{2} \text{ area BOC},$$
$$\cos ZX = \cos \tfrac{1}{2} CA \,.\, \cos \tfrac{1}{2} \text{ area COA},$$
$$\cos XY = \cos \tfrac{1}{2} AB \,.\, \cos \tfrac{1}{2} \text{ area AOB},$$

therefore, if two of the three arcs YZ, ZX, XY be given, the corresponding two of the three areas BOC, COA, AOB are given, and therefore the point O is given as the intersection of two well-known circles.

2914. (Proposed by J. J. WALKER, M.A.)—If ABC be a triangle inscribed in, and concentroidal with, an ellipse whose semi-axes are a, b; prove that (1) the sum of the squares of the sides of such a triangle is constant, and equal to $\frac{9}{2}(a^2+b^2)$; (2) the sum of the squares of the lines joining A, B, C with the fourth point D, in which the circumscribing circle cuts the ellipse, is equal to $3\left\{\frac{1}{2}(a^2+b^2)+r^2\right\}$, r being the semi-diameter drawn to the point; and (3) the sum of the squares of the diameters of the ellipse passing through the points A, B, C, is equal to the sum of the squares of three conjugate diameters.

I. *Solution by* JAMES DALE, *and others.*

Since the diameters AA′, BB′, CC′ respectively bisect BC, CA, AB; Oa, Ob, Oc, which are parallel to BC, CA, AB, are conjugate to OA, OB, OC respectively. By the property of the centre of gravity, $BC^2+CA^2+AB^2 = 3(OA^2+OB^2+OC^2)$; and since BC passes through the middle point of OA′, and is parallel to OA, $BC^2 = 3Oa^2$; similarly, $CA^2 = 3Ob^2$, $AB^2 = 3Oc^2$;

$$\therefore\ 2(BC^2+CA^2+AB^2) = 3\left\{(OA^2+Oa^2)+(OB^2+Ob^2)+(OC^2+Oc^2)\right\}$$
$$= 9(a^2+b^2),$$

and
$$OA^2+OB^2+OC^2 = Oa^2+Ob^2+Oc^2,$$

which proves (1) and (3). To prove (2), let (x_1, y_1), (x_2, y_2), (x_3, y_3), (h, k) be the coordinates of A, B, C, D; then

$$AD^2 = (h-x_1)^2+(k-y_1)^2 = r^2+OA^2-2hx_1-2ky_1;$$

so also $BD^2 = r^2+OB^2-2hx_2-2ky_2$, $CD^2 = r^2+OC^2-2hx_3-2ky_3$;

therefore $AD^2+BD^2+CD^2 = 3r^2+OA^2+OB^2+OC^2 = 3\left\{\frac{1}{2}(a^2+b^2)+r^2\right\}$,

because
$$x_1+x_2+x_3 = 0,\quad y_1+y_2+y_3 = 0.$$

II. *Solution by* STEPHEN WATSON.

Put a', b' for the semi-axes of the given ellipse; then, taking BC, BA as axes of reference, the equation of the ellipse concentroidal with the triangle

is
$$c^2x^2+caxy+a^2y^2-ca(cx+ay) = 0 \quad\text{.........................}(\alpha),$$

its centre being at the point $O = (\frac{1}{3}a, \frac{1}{3}c)$, and a line through O parallel to the tangent to (α) at the origin is

$$y-\tfrac{1}{3}c = -\frac{c}{a}(x-\tfrac{1}{3}a).$$

This meets (1) in the point $B' = \left\{\frac{1}{3}a(1+\sqrt{3}),\ \frac{1}{3}c(1+\sqrt{3})\right\}$;

hence $OB'^2 = \frac{1}{3}a^2+\frac{1}{3}c^2-\frac{2}{3}ac\cos B = \frac{1}{3}b^2$,

also $OB^2 = \frac{1}{9}a^2+\frac{1}{9}c^2+\frac{2}{9}ac\cos B = \frac{2}{9}a^2+\frac{2}{9}c^2-\frac{1}{9}b^2$;

therefore $a'^2+b'^2 = OB^2+OB'^2 = \frac{2}{9}(a^2+b^2+c^2)$, which proves (1).

Denote the point D by (x_1, y_1); then

$$AD^2 = x_1^2+(y_1-c)^2+2x_1(y_1-c)\cos B,$$
$$BD^2 = x_1^2+y_1^2+2x_1y_1\cos B,$$
$$CD^2 = (x_1-a)^2+y_1^2+2(x_1-a)y_1\cos B,$$
$$r^2 = (x_1-\tfrac{1}{3}a)^2+(y_1-\tfrac{1}{3}c)^2+2(x_1-\tfrac{1}{3}a)(y_1-\tfrac{1}{3}c)\cos B,$$

therefore $AD^2+BD^2+CD^2-3r^2 = \frac{2}{3}(a^2+c^2-ac\cos B)$
$$= \tfrac{1}{3}(a^2+b^2+c^2) = \tfrac{2}{3}(a'^2+b'^2),$$
which proves (2).

Again, OB′ is the semi-conjugate to OB, and let OA′, OC′ be those to OA, OC; then, as OB'^2, OB^2 were found above, we have

$$OA'^2 = \tfrac{1}{3}a^2,\quad OC'^2 = \tfrac{1}{3}c^2,\quad OA^2 = \tfrac{2}{9}(b^2+c^2)-\tfrac{1}{9}a^2,\quad OC^2 = \tfrac{2}{9}(a^2+b^2)-\tfrac{1}{9}c^2,$$

therefore $OA'^2+OB'^2+OC'^2 = \frac{1}{3}(a^2+b^2+c^2) = OA^2+OB^2+OC^2$,
which proves (3).

III. *Solution by the* Proposer.

1. The coordinates of A, B, C are the roots of the equations
$$4x^3-3a^2x-a^2x' = 0,\quad 4y^3-3b^2y-b^2y' = 0 \;\ldots\ldots\ldots\ldots\ldots(\alpha,\ \beta),$$
(x', y') being the fourth point in which the circle circumscribing the triangle ABC meets the ellipse. Now the sum of the squares of the differences of the roots of (α) is equal to $\frac{9}{4}a^2$; and the same function of the roots of (β) is equal to $\frac{9}{4}b^2$; whence $AB^2+BC^2+CA^2 = \frac{9}{4}(a^2+b^2)$.

2. Again, if $\alpha, \alpha', \alpha''$ be the roots of (α), and β, β', β'' those of (β),
$$\Sigma(x'-\alpha)^2 = 3x'^2-2x'\Sigma\alpha+\Sigma\alpha^2 = 3(\tfrac{1}{2}a^2+x'^2);$$
similarly $\qquad \Sigma(y'-\alpha)^2 = 3(\tfrac{1}{2}b^2+x'^2);$

whence $\qquad AD^2+BD^2+CD^2 = 3\left\{\tfrac{1}{2}(a^2+b^2)+r^2\right\}.$

3. If O be the centre of the ellipse,
$$OA^2+OB^2+OC^2 = \Sigma\alpha^2+\Sigma\beta^2 = \tfrac{3}{2}(a^2+b^2);$$
but the six squares of these semi-diameters and the conjugates $= 3(a^2+b^2)$.

2913. (Proposed by R. Tucker, M.A.)—Straight lines are drawn from the angles A, B, C of the triangle ABC, through a point O within it, meeting the opposite sides in D, E, F; show that the locus of O, when the perpendiculars from A, B, C upon EF, FD, DE meet in a point, is a cubic through A, B, C. Find also the locus of the latter point.

I. *Solution by* Stephen Watson.

Let the equations of AD, BE, CF be $l\alpha = m\beta = n\gamma$ ……………………(1);
then that of EF is $l\alpha+m\beta-n\gamma = 0$, and a perpendicular to this from A is
$$\frac{\beta}{\gamma} = \frac{m+l\cos C-n\cos A}{n-m\cos A+l\cos B}\;\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Similarly,

$$\frac{\gamma}{\alpha} = \frac{n+m\cos A - l\cos B}{l-n\cos B+m\cos C}, \quad \text{and} \quad \frac{\alpha}{\beta} = \frac{l+n\cos B - m\cos C}{m-l\cos C+n\cos A} \quad \text{.........(3, 4)}$$

are the equations of perpendiculars from B and C to FD, DE. When (2), (3), (4) meet in a point, the product of the right-hand members must $=1$; and eliminating from the result thus obtained the quantities l, m, n, by means of (1), we get, after reduction, for the locus of O the equation

$$\sin^2 A\,(\beta\cos B - \gamma\cos C)\,\alpha^2 + \sin^2 B\,(\gamma\cos C - \alpha\cos A)\,\beta^2 + \sin^2 C\,(\alpha\cos A - \beta\cos B)\,\gamma^2 = 0 \quad \text{.................... (5).}$$

The equations (2), (3), (4) may be written

$$(\beta+\gamma\cos A)\,n - (\gamma+\beta\cos A)\,m + (\beta\cos B - \gamma\cos C)\,l = 0 \quad \text{.........(6),}$$

$$-(\alpha+\gamma\cos B)\,n + (\gamma\cos C - \alpha\cos A)\,m + (\gamma+\alpha\cos B)\,l = 0 \quad \text{.........(7),}$$

$$(\alpha\cos A - \beta\cos B)\,n + (\alpha+\beta\cos C)\,m - (\beta+\alpha\cos C)\,l = 0 \quad \text{.........(8),}$$

and $\alpha\,(6)+\beta\,(7)-\gamma\,(8)$ gives $\dfrac{l}{m} = \dfrac{(\gamma+\beta\cos A)\,\alpha}{(\gamma+\alpha\cos B)\,\beta}$.

Similarly, $\dfrac{m}{n} = \dfrac{(\alpha+\gamma\cos B)\,\beta}{(\alpha+\beta\cos C)\,\gamma}$, $\dfrac{n}{l} = \dfrac{(\beta+\alpha\cos C)\,\gamma}{(\beta+\gamma\cos A)\,\alpha}$;

hence, eliminating l, m, n, we have, from the locus of the "latter point," the equation

$$(\cos A - \cos B\cos C)\,(\beta^2-\gamma^2)\,\alpha + (\cos B - \cos C\cos A)\,(\gamma^2-\alpha^2)\,\beta + (\cos C - \cos A\cos B)\,(\alpha^2-\beta^2)\,\gamma = 0 \quad \text{.....................(9).}$$

Putting successively $\alpha, \beta, \gamma = 0$, in the cubics (5) and (9), the results give

respectively $\dfrac{\sin^2 A}{\cos A}\,\alpha = \dfrac{\sin^2 B}{\cos B}\,\beta = \dfrac{\sin^2 C}{\cos C}\,\gamma$,

$$\frac{\alpha}{\cos A - \cos B\cos C} = \frac{\beta}{\cos B - \cos C\cos A} = \frac{\gamma}{\cos C - \cos A\cos B};$$

hence, in each case, the cubic meets the sides of the triangle in three points such that, when joined to the opposite angles, the three joining lines are concurrent.

II. *Solution by* JAMES DALE, *and others.*

Let AO, BO, CO meet the opposite sides in D, E, F respectively; then, if f, g, h are the coordinates (trilineal) of O, the equations of EF, FD, DE

are $\quad -\dfrac{x}{f}+\dfrac{y}{g}+\dfrac{z}{h}=0,\quad \dfrac{x}{f}-\dfrac{y}{g}+\dfrac{z}{h}=0,\quad \dfrac{x}{f}+\dfrac{y}{g}-\dfrac{z}{h}=0;$

and the equations to the perpendiculars from A, B, C are

$$(\;gh\cos B - hf\cos A + fg)\,y - (\;gh\cos C + hf - fg\cos A)\,z = 0,$$

$$(\;gh + hf\cos C - fg\cos B)\,z - (-gh\cos B + hf\cos A + fg)\,x = 0,$$

$$(-gh\cos C + hf + fg\cos A)\,x - (\;gh - hf\cos C + fg\cos B)\,y = 0.$$

The condition that these lines should pass through one point is

$$(-gh\cos C + hf + fg\cos A)(gh\cos B - hf\cos A + fg)(gh + hf\cos C - fg\cos B)$$
$$-(-gh\cos B + hf\cos A + fg)(gh - hf\cos C + fg\cos B)(gh\cos C + hf - fg\cos A)$$
$$= 0,$$

which, being simplified, shows that (f, g, h) lies on the cubic

$$x\cos A(y^2\sin^2 B - z^2\sin^2 C) + y\cos B(z^2\sin^2 C - x^2\sin^2 A)$$
$$+ z\cos C(x^2\sin^2 A - y^2\sin^2 B) = 0,$$

or $a^2x^2(y\cos B - z\cos C) + b^2y^2(z\cos C - x\cos A) + c^2z^2(x\cos A - y\cos B) = 0$.

This cubic passes through the angular points of the triangle, through the intersection of the perpendiculars, and also through the four points

$$\pm ax = \pm by = \pm bz.$$

Eliminating f, g, h from the equations to the perpendiculars, we find the locus of the intersection of these perpendiculars

$$x(y^2 - z^2)(\cos A - \cos B\cos C) + y(z^2 - x^2)(\cos B - \cos C\cos A)$$
$$+ z(x^2 - y^2)(\cos C - \cos A\cos B) = 0.$$

This cubic also passes through the angular points of the triangle, through the centre of the circumscribing circle, and through the centres of inscribed and escribed circles, and through the intersection of the perpendiculars.

2924. (Proposed by the Rev. J. Wolstenholme, M.A.)—On a focal chord PSQ of a parabola are taken p, q, on opposite sides of S, such that $Sp \cdot Sq = SP \cdot SQ$, and any parabola is described through p, q, and having its axis parallel to that of the former: prove that their chord of intersection will pass through S.

I. *Solution by* W. K. Clifford, B.A.

I consider the following more general question :—

Through a point a let a line B be drawn meeting a conic in l, m; then the quantity $al \cdot am \sin BP \cdot \sin BQ$ ....(1)
(where P and Q are the asymptotes) is independent of the position of the line B, and may be called the *distance* of the point a from the conic. What now is the locus of a point equidistant from two given conics?

Let $C_2 = 0$, $D_2 = 0$ be the equations of the conics, and let a^2C_2 denote the result of substituting the coordinates of the point a for the variables in C_2; also let i, j be the circular points at infinity. Then I find that the distance (above defined) of the point a from C_2 is

$$\frac{a^2C_2}{(aij)^2 \cdot \sqrt{(i^2C_2 \cdot j^2C_2)}} \quad \text{..............................(2)},$$

where, of course, (aij) means the determinant formed with the coordinates of the points a, i, j.

This being so, the equation of the required locus is

$$\frac{C_2}{\sqrt{(i^2C_2 \cdot j^2C_2)}} = \frac{D_2}{\sqrt{(i^2D_2 \cdot i^2D_2)}} \quad \text{....................(3)},$$

showing that the locus is a conic passing through the intersections of the two given ones.

Now if we are using Cartesian coordinates, and if the two conics are similar and similarly situated, it is easy to see that the terms of the second order have entirely disappeared from the equation (3); which indicates that the line at infinity is part of the locus. The remainder of it is then their finite chord of intersection; which is a true radical axis, in the sense that if any line whatever is met by the radical axis in a, by C_2 in l, m, and by D_2 in l', m', we must have always

$$al \,.\, am = al' \,.\, am' \qquad\qquad (4).$$

To apply this to Quest. 2924, we have only to observe that parabolæ with parallel axes are homothetic, or similar and similarly situated curves; and that the equation $Sp \,.\, Sq = SP \,.\, SQ$ indicates that the focus is situated on their radical axis.

The theorem of the radical axis of two homothetic conics may of course be proved for ellipses by orthogonal projection from the circle, and then extended by the doctrine of continuity to the rest.

II. *Solution by* WILLIAM ROBERTS, JUN.

Let the equation of the parabola be $y^2-px=0$, and that of any parabola having its axis parallel to that of the former $y^2-p'x+2fy+c=0$. Then it is easy to see that

$$SP \,.\, SQ = \frac{-p^2}{4\sin^2\theta}, \qquad Sp \,.\, Sq = \frac{-pp'-4c}{4\sin^2\theta},$$

θ being the direction of the chord. Therefore $c = \frac{1}{4}p\,(p'-p)$. But this is the condition that the chord should pass through S.

2806. (Proposed by F. D. THOMSON, M.A.)—L, M, N are the real points of inflexion of a cubic; PQR is the triangle formed by the tangents at L, M, N; p, q, r are the points forming with L, M, N harmonic sections on the sides of PQR. Prove that the Hessian of the cubic touches the triangle PQR at the points p, q, r. Also, if P′Q′R′ be the triangle formed by the tangents to the Hessian at L, M, N, then PP′, QQ′, RR′ meet in the point of intersection of the harmonic polars of L, M, N.

I. *Solution by* JAMES DALE.

Taking PQR as triangle of reference, the equation to the cubic may be put in the form $(lx+my+nz)^3+6kxyz=0$, the points L, M, N being respectively $(x=0,\ my+nz=0)$, $(y=0,\ nz+lx=0)$, $(z=0,\ lx+my=0)$; and the equation to the Hessian is

$$(lx+my+nz)\,(l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy)+kxyz=0,$$

the form of which shows that it touches the sides of the triangle of reference in the points

$$(x=0,\ my-nz),\quad (y=0,\ nz-lx=0),\quad (z=0,\ lx-my=0),$$

which are the harmonic conjugates of L, M, N respectively.

The tangents to the Hessian at L, M, N are respectively

$$4mn\,(lx+my+nz)-kx=0\,;\qquad 4nl\,(lx+my+nz)-ky=0\,;$$
$$4lm\,(lx+my+nz)-kz=0\,;$$

and hence are obtained the equations of PP′, QQ′, RR′, viz. $my-nz=0$, $nz-lx=0$, $lx-my=0$, intersecting in $lx=my=nz$, which is also the point of intersection of Pp, Qq, Rr.

II. *Solution by the* Proposer.

1. It may be shown that, if the equation to a curve be given under the form $U\equiv\phi(x, y, z, t)=0$, where x, y, z, t are connected by the identical relation $$ax+by+cz+dt=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$ that the polar conic of a point (x, y, z, t) on the curve has for its equation

$$S\equiv(A, B, C, D, E, F, G, H, K, L\between\bar{x}, \bar{y}, \bar{z}, \bar{t})^2=0,$$

where $\quad A=\dfrac{d^2U}{dx^2},\ \ldots\ldots\quad E=\dfrac{d^2U}{dy\,dz},\ \ldots\ldots\quad H=\dfrac{d^2U}{dx\,dt},\ \ldots\ldots$

Now for a point of inflexion S becomes two straight lines, and it may be shown that the discriminant of S is given by

$$\Delta\equiv\begin{vmatrix} A, & G, & F, & H, & a\\ G, & B, & E, & K, & b\\ F, & E, & C, & L, & c\\ H, & K, & L, & D, & d\\ a, & b, & c, & d, & 0\end{vmatrix}$$

Hence, by definition, the equation to the Hessian is $\Delta=0$.

2. Now the equation to the cubic may be written $3xyz+t^3=0\equiv U$, t being the line LMN, and x, y, z the three tangents at L, M, N. Here $A=B=C=0$, $D=6t$, $E=3x$, $F=3y$, $G=3z$, $H=K=L=0$. Hence the equation to the Hessian is

$$H(U)\equiv\begin{vmatrix} 0\ , & 3z, & 3y, & 0\ , & a\\ 3z, & 0\ , & 3x, & 0\ , & b\\ 3y, & 3x, & 0\ , & 0\ , & c\\ 0\ , & 0\ , & 0\ , & 6t, & d\\ a\ , & b\ , & c\ , & d\ , & 0\end{vmatrix}=0,$$

which reduces to

$$H(U)\equiv t\left\{a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy\right\}-d^2xyz=0,$$

which is satisfied by $x=0$, $t=0$, and $x=0$, $(by-cz)^2=0$. Hence H (U) touches PQR in the points p, q, r.

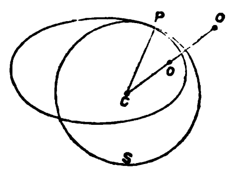
3. Also the equations to the tangents to H (u) at L, M, N are

$d^2x+4bct=0$ (Q′R′), $\quad d^2y+4cat=0$ (R′P′), $\quad d^2z+4abt=0$ (P′Q′).

Hence PP′, QQ′, RR′ meet in the point $ax=by=cz$.

2818. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Given a circle S and a straight line α not meeting S in real points; O, O′ are the two-point circles to which, and S, α is the radical axis; two conics are drawn osculating S in the same point P, and having one focus at O, O′ respectively: prove that the corresponding directrices coincide.

Solution by MATTHEW COLLINS, B.A.



Let C be the centre of the given circle S; P the origin, the axes of x and y being PC and the tangent at P; then the equation of the given circle whose radius (PC) is r will be

$$x^2+y^2-2rx=0 \text{ .... } (1).$$

Now this circle has plainly a three-point osculation at P with the curve

$$y^2-2rx+Ax^2+Bxy+Dx^3+Ey^3+Fx^2y\ldots=0,$$

since both equations give the same value for $\dfrac{y^2}{x}$ at or rather near P, where x and y are infinitely small, and where therefore x^2, xy, xy^2, y^3, &c. can and should be rejected as infinitely small, compared to the terms retained, viz., those containing x and y^2; we may, therefore, conveniently represent the conic osculating the circle S at P by

$$(a^2-1)\,x^2+2abxy+(b^2-1)\,y^2-2rx\,(b^2-1)=0 \text{ .... } (2),$$

which, on account of a and b being indeterminate, can be made to pass through two other given points, or to have a given focus, &c.

Now (2) gives $\qquad (ax+by)^2=x^2+y^2+2rx\,(b^2-1),$

or, therefore, $\;(ax+by+c)^2=x^2+y^2+2acx+2bcy+2rx\,(b^2-1)+c^2$

say $\qquad\qquad =(x+m)^2+(y+n)^2 \text{ .... } (3);$

and in this form (3) plainly represents a conic, the coordinates of whose focus are $(-m,\ -n)$, and the equation of whose directrix is

$$ax+by+c=0 \text{ .... } (4).$$

Now (3) gives $\;m^2+n^2=c^2,\; n=bc,\; m=ac+r\,(b^2-1)$ .... (5);

these easily give $\;b^2=\dfrac{n^2}{m^2+n^2}\;$ and $\;ac=m+r-b^2r=m+r-\dfrac{rn^2}{m^2+n^2},$

and thence (4) $\times\,\dfrac{c}{n}$ gives

$$\left(\frac{m+r}{n}-\frac{rn}{m^2+n^2}\right)x+y+\frac{m^2+n^2}{n}=0 \text{ .... } (6).$$

Now this equation of the directrix will not change even if m and n (and, therefore, the focus O) do change, *provided that* $\dfrac{m^2+n^2}{n}$ remains constant, say $=f$ .... (7), and *also that* $\dfrac{m+r}{n}$ remains constant, say $=g$...... (8).

Now (8) indicates that the point (m, n), that is the focus O, in changing must move upon a straight line passing through the centre C, and (7) indi-

cates that the same point must still lie upon a circle touching PC at P; and hence, if the focus O move *without changing* its corresponding directrix, it can only move to O′, the point *inverse* to O with respect to circle S, COO′ being one straight line, and CO . CO′ = CP^2.

[A Solution by Mr. TUCKER is given on p. 74 of Vol. XI. of the *Reprint*.]

2873. (Proposed by J. J. WALKER, M.A.)—1. Determine the point D in any quadrant of an ellipse, such that if the osculating circle at it meet the ellipse again in C, that at C will pass through D.

2. Show that A, B, the other two points on the ellipse, the osculating circles at which also pass through D, are the ends of two semi-diameters, each conjugate to that semi-diameter lying in D's quadrant which is equal to the other.

Solution by JAMES DALE.

1. Since the osculating circle at D passes through C, and that at C through D, CD is the common chord of these two circles; and, making equal angles with the tangents at C and D, these tangents must be parallel, and therefore CD must pass through the centre. If (h, k) be the coordinates of D, the equation to the common chord is $y-k = \frac{b^2h}{a^2k}(x-h)$, which in this case must be satisfied by $x=0$, $y=0$; therefore $a^2k^2-b^2h^2=0$; or, there are four points which fulfil the required condition, viz. the extremities of equal conjugate diameters.

2. Taking $x = \frac{1}{2}a\sqrt{2}$, $y=\frac{1}{2}b\sqrt{2}$ as the coordinates of D, the coordinates of A or B (h, k) are determined by the condition that the line AD or BD makes the same angle with the axis that the tangent at A or B makes;

therefore $$\frac{k-\frac{1}{2}b\sqrt{2}}{h-\frac{1}{2}a\sqrt{2}} = \frac{b^2h}{a^2k},$$

whence $$(a^2k^2-b^2h^2)\sqrt{2} = ab\,(ak-bh), \quad (ak+bh)\sqrt{2} = ab;$$

therefore $$2a^2k^2+4abhk+2b^2h^2 = a^2b^2 = a^2k^2+b^2h^2;$$

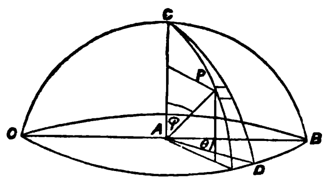
therefore $$a^2k^2+4abhk+b^2h^2 = 0;$$

and if (θ_1, θ_2) be the angles that AD, BD make with the axis, we have from the above equation $\tan\theta_1 . \tan\theta_2 = \frac{b^2}{a^2}$, which proves (2).

2889. (Proposed by J. J. WALKER, M.A.)—The particles of a thin hemispherical shell being supposed to attract with a force varying directly as the distance, prove that the resultant attraction on any point of the rim is the same as if all the matter of the shell were concentrated at the middle point of its axis.

Solution by F. D. THOMSON, M.A.; W. ROBERTS; *the* PROPOSER; *and others.*

Let a be the radius of the shell; then, taking the usual angles at the centre as variables, element of surface at $\mathrm{P} = a^2 \sin\phi\, d\phi\, d\theta$, and attraction to $\mathrm{O} = \mu \mathrm{OP} a^2 \sin\phi\, d\phi\, d\theta$; therefore attraction of element in direction OA



$$= \mu a^3 (1 + \sin\phi \cos\theta) \sin\phi\, d\phi\, d\theta;$$

therefore, if X be the total attraction in this direction,

$$\mathrm{X} = 2\mu a^3 \int_0 \int_0^{\pi} (\sin\phi + \sin^2\phi \cos\theta)\, d\phi\, d\theta = 2\mu a^3 \pi \int_0^{\frac{1}{2}\pi} \sin\phi\, d\phi = 2\mu\pi a^3.$$

Similarly, if Z be the total attraction parallel to AC,

$$\mathrm{Z} = 2\mu a^3 \int_0^{\frac{1}{2}\pi} \int_0^{\pi} \cos\phi \sin\phi\, d\phi\, d\theta = \mu\pi a^3 \int_0^{\frac{1}{2}\pi} \sin 2\phi\, d\phi = \mu\pi a^3;$$

therefore $\dfrac{\mathrm{Z}}{\mathrm{X}} = \dfrac{1}{2}$. Therefore the resultant attraction is towards the middle point of AC, and is equal to $\mu\pi a^3 \sqrt{5}$.

But the distance of this point from O is equal to $\frac{1}{2} a \sqrt{5}$, and whole surface of shell $= 2\pi a^2$; therefore total attraction $= \mu \,.\, \frac{1}{2} a \sqrt{5} \times 2\pi a^2$.

[Mr. TUCKER, J. DALE, and others remark that the theorem follows at once from the general property, that "if the particles of a body attract with a force varying as product of the mass into the distance, the resultant attraction is the same as if the whole mass of the body were collected at its centre of gravity" (Todhunter's *Analytical Statics*, p. 246, Cor. 1), which, in this case, is the middle point of the axis.]

2869. (Proposed by J. J. WALKER, M.A.)—1. The square of the chord common to an ellipse and its circle of curvature at any point is equal to $\dfrac{16}{(a^2 - b^2)^2} (a^2 - r'^2)(r'^2 - b^2) r'^2$, where a, b are the semi-axes, and r' the semi-diameter parallel to the tangent at the point.

2. When the chord is a maximum, $3r'^2 = a^2 + b^2 + (a^4 - a^2 b^2 + b^4)^{\frac{1}{2}}$.

3. In the parabola, the square of the common chord is equal to $16p'x$, p' being the parameter, and x the abscissa of the point of contact.

I. *Solution by* JAMES DALE.

1. The equation of a chord common to an ellipse and its osculating circle at the point (h, k) is $b^2 h (x - h) - a^2 k (y - k) = 0$; and if (h_1, k_1) are the coordinates of the point where the common chord again meets the ellipse, we readily find $$h - h_1 = \frac{4k^2 h}{b^2}, \quad k - k_1 = \frac{4h^2 k}{a^2};$$

therefore $$(\text{chord})^2 = 16h^2k^2\left(\frac{h^2}{a^4}+\frac{k^2}{b^2}\right),$$

which, by means of the equation $r'^2 = a^2+b^2-(h^2+k^2)$, becomes

$$(\text{chord})^2 = \frac{16}{(a^2-b^2)^2}(a^2-r'^2)(r'^2-b^2)r'^2.$$

2. Differentiating with respect to r', and putting the result $=0$, we get $3r'^4-2r'^2(a^2+b^2)+a^2b^2=0$; therefore $r'^2=\frac{1}{3}\left\{a^2+b^2\pm\sqrt{(a^4-a^2b^2+b^4)}\right\}$.

The negative sign makes $r'^2<b^2$, which is impossible; therefore the maximum value of the chord is given by

$$r'^2=\tfrac{1}{3}\left\{a^2+b^2+\sqrt{(a^4-a^2b^2+b^4)}\right\}.$$

3. The equation to the common chord in the parabola $y^2=4ax$ is

$$y-k=-\frac{2a}{12}(x-h),\quad\text{or}\quad y=\frac{2a(3h-x)}{k};$$

and if (h_1, k_1) be the coordinates of the point where the chord again meets the parabola, we have $h_1=9h$, and $k_1=-3k$;

therefore $(\text{chord})^2=(h-h_1)^2+(k-k_1)^2=64h^2+16k^2=64h^2+64ah$
$=64a(a+h)=16p'$. (focal rad. vect. of point of contact).

II. *Solution by the* Proposer.

1. If (x, y) be the point of contact, and (α, β) the other end of the chord in question, it is known that

$$4x^3-3a^2x-a^2\alpha=0,\quad\text{or}\quad x-\alpha=4\frac{a^2-x^2}{a^2}x=4\frac{xy^2}{b^2}.$$

Similarly $y-\beta=4\dfrac{x^2y}{a^2}$, whence $(x-\alpha)^2+(y-\beta)^2=\dfrac{16}{a^4b^4}(b^4x^2+a^4y^2)x^2y^2$,

which is the value of the square of the chord in terms of the coordinates of the point of contact.

But $b^4x^2+a^4y^2=a^2b^2r'^2$, and $x^2=\dfrac{a^2(a^2-r'^2)}{a^2-b^2}$, $y^2=\dfrac{b^2(b^2-r'^2)}{b^2-a^2}$.

2. When the chord is a maximum,

$$(a^2-r'^2)(r'^2-b^2)+(a^2-r'^2)r'^2-(r'^2-b^2)r'^2=0,$$

or $$3r'^4-2(a^2+b^2)r'^2+a^2b^2=0,$$

whence $$r'^2=\tfrac{1}{3}\left\{a^2+b^2+(a^4-a^2b^2+b^4)^{\frac{1}{2}}\right\}.$$

[That the positive sign should be taken, appears from this: if the negative were taken, it would follow that

$$3r^2=2(a^2+b^2)+\left\{(a^2-b^2)^2+a^2b^2\right\}^{\frac{1}{2}},\quad i.\,e.,\ 3r_{\cdot}^2>3a^2+b^2.]$$

3. In the case of the parabola $y^2=px$, it is easily found that

$$\beta=-3y\ \text{ or }\ \alpha=9x,\quad\text{whence}\quad \beta-y=-4y,\ \alpha-x=8x,$$

and $$(\alpha-x)^2+(\beta-y)^2=16(y^2+4x^2)=16(p+4x)x=16p'x.$$

2920. (Proposed by Professor CAYLEY.) — Imagine a tetrahedron $BB'CC'$ in which the opposite sides BB', CC' are at right angles to each other and to the line joining their middle points M, N; and in which moreover $\overline{CN}^2+\overline{NM}^2+\overline{MB}^2=0$, (or, what is the same thing, the sides CB, CB', $C'B$, $C'B'$ are each $=0$; the tetrahedron is of course imaginary; viz., the lines CC', BB' and points M, N may be real; but the distances $MB = MB'$ and $NC = NC'$ may be one real and the other imaginary, or both imaginary, but they cannot be both real) the points B, B′ and C, C′ are said to be "skew antipoints." Then it is required to prove that

1. A given system of skew antipoints may be taken to be the nodes (conical points) of a tetranodal cubic surface, passing through the circle at infinity, and which is in fact a Parabolic Cyclide.

2. The equation of the surface may be expressed in the form

$$x(x+\beta)(x+\gamma)+(x+\beta)y^2+(x+\gamma)z^2=0.$$

3. The section through either of the lines ($y=0$, $x+\gamma=0$) and ($z=0$, $x+\beta=0$) is made up of this line and a circle; the two systems of circles being the curves of curvature of the surface; it is required to verify this *à posteriori*; viz., by means of the equation of the surface to transform the differential equation of the curves of curvature in such manner that the transformed equation shall have the integrals

$$y = C(x+\gamma),\quad z = C'(x+\beta).$$

Solution by the Rev. J. WOLSTENHOLME, M.A.

To find the conical points of the surface

$$x(x+\beta)(x+\gamma)+y^2(x+\beta)+z^2(x+\gamma)=0,$$

we have the three equations

$$(x+\beta)(x+\gamma)+x(2x+\beta+\gamma)+y^2+z^2=0,\quad y(x+\beta)=0,\quad z(x+\gamma)=0,$$

which give the four points

$$(x+\beta)=0,\ z=0,\ y^2=\beta(\gamma-\beta);\quad x+\gamma=0,\ y=0,\ z^2=\gamma(\beta-\gamma);$$

and these points lie on the surface. If these points be B, B′, C, C′, and M, N middle points of BB', CC', we have

$$BM^2+MN^2+NC^2=\beta(\gamma-\beta)+(\beta-\gamma)^2+\gamma(\beta-\gamma)\equiv 0 \quad\text{or}\quad BC=0,$$

and similarly BC', $B'C$, $B'C'$ each $=0$. The points will be two real and two impossible if β, γ have like signs, and will be all impossible if β, γ have unlike signs. The terms of highest dimensions in the equation are $x(x^2+y^2+z^2)$, and therefore the surface contains the circle at infinity.

The equation of the normal at any point (X, Y, Z) of the surface is

$$\frac{x-X}{(X+\beta)(X+\gamma)+X(2X+\beta+\gamma)+Y^2+Z^2}=\frac{y-Y}{2Y(X+\beta)}=\frac{z-Z}{2Z(X+\gamma)};$$

and where this meets the plane of xy,

$$z=0,\quad y=Y\left(1-\frac{X+\beta}{X+\gamma}\right)=\frac{Y(\gamma-\beta)}{X+\gamma},$$

$$\begin{aligned}2x &= 2X-(X+\beta)-X-\frac{X(X+\beta)}{X+\gamma}-\frac{Y^2+Z^2}{X+\gamma}\\ &= -\beta-\frac{X(X+\beta)}{X+\gamma}-\frac{Y^2(X+\gamma)-Y^2(X+\beta)-X(X+\beta)(X+\gamma)}{(X+\gamma)^2};\end{aligned}$$

by equation of the surface, $\quad = -\dfrac{Y^2(\gamma-\beta)}{(X+\gamma)^2}-\beta;$

therefore $\quad y^2 = (\beta-\gamma)(2x+\beta).$

So where the normal meets the plane of xz, $y=0$, $z^2=(\gamma-\beta)(2x+\gamma)$, or the normal meets the planes of xy, xz in two fixed parabolas, either of which is the locus of the vertices of all cones of revolution containing the other. The surface is then a Parabolic Cyclide, and its lines of curvature are circles. This surface is the locus of the points of contact of tangent planes to a series of confocal conicoids drawn from a fixed point in one of the axes. For if

$$\frac{x^2}{k}+\frac{y^2}{k-b^2}+\frac{z^2}{k-c^2}=1$$

be one of the conicoids, and tangent planes be drawn through the point $(h, 0, 0)$, we have for the points of contact $\dfrac{hx}{k}=1$, and the equation of the locus for different values of k is

$$\frac{x}{h}+\frac{y^2}{hx-b^2}+\frac{z^2}{hx-c^2}=1,$$

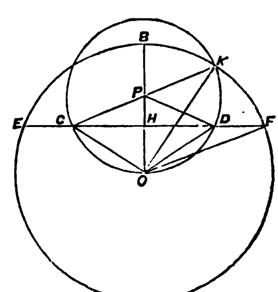
or, removing the origin to the point $(h, 0, 0)$, and altering the notation,

$$x+\frac{y^2}{x-\beta}+\frac{z^2}{x-\gamma}=0.$$

2296. (Proposed by Professor SYLVESTER.)—If a triangle be formed with the centre of a circle as vertex, and the line joining two points taken at random within the circle as base; show that the chances of the angle at the centre being the greatest, intermediate, or least angle of the triangle, are respectively as

$$4\pi+6\sqrt{3} : 4\pi-3\sqrt{3} : 4\pi-3\sqrt{3}.$$

Solution by STEPHEN WATSON.



Let O be the centre of the circle, OB a radius, P any point in OB, OCPD a rhombus having each side equal to OP. Through C, D draw the chord EF; with P as centre and radius PO describe a circle cutting the given circle in K; and join OK. Put OB$=a$, OP$=x=2$OH. Then in the triangle OPQ the angle O will be the greatest, when Q lies anywhere below EF but without the circle COD, and this area is

$$a^2\left\{\frac{\pi}{2}+\sin^{-1}\frac{x}{2a}\right\}+\tfrac{1}{2}x\left(a^2-\tfrac{1}{4}x^2\right)^{\frac{1}{2}}$$
$$-\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)x^2;$$

hence, multiplying by $2\pi x dx$, integrating between $x=0$, $x=a$, and dividing by $a^4\pi^2$, we get $$p_1 = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} \text{ .... } (1).$$

When O is the least angle, Q must lie above EF, in the area common to the circles EBF, COG; and when $x > \frac{1}{2}a$, this area is

$$a^2\cos^{-1}\frac{a}{2x} - x^2\left(\pi - 2\cos^{-1}\frac{a}{2x}\right) - a\left(x^2 - \tfrac{1}{4}a^2\right)^{\frac{1}{2}} - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)x^2 \text{......}(2);$$

but when $x < \frac{1}{2}a$, the area is $\left(\frac{2}{3}\pi + \frac{\sqrt{3}}{4}\right)x^2$ ....(3);

therefore $$p_3 = \frac{1}{a^4\pi^2}\left\{\int_{\frac{1}{2}a}^{a}(1) + \int_{0}^{\frac{1}{2}a}(2)\right\}2\pi x dx = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \text{ .... } (4);$$

and $$p_2 = 1 - p_1 - p_3 = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \text{ .... } (5);$$

therefore $$p_1 : p_2 : p_3 = 4\pi + 6\sqrt{3} : 4\pi - 3\sqrt{3} : 4\pi - 3\sqrt{3}.$$

Hints on Gravitation. *By* Septimus Tebay, B.A.

Has matter an innate power of attraction? Of all forces with which we are acquainted, attraction is the only force which is presumed to act on a distant body without a connection. Is this force real, or only apparent? It is now generally admitted that light is produced by the vibrations of an elastic fluid which fills all space. If a small portion of this fluid were annihilated, or the pressure about a point suddenly diminished, there would be a general tendency towards this point from all parts of the surrounding fluid. Let two material particles be placed in this fluid at rest, without weight, or the power of attraction, and acted on by no forces. They will remain at rest for ever. Let us see what would be the consequence if one of them were suddenly set in motion. If the particle be very small, the fluid will not be sensibly disturbed at a considerable distance from the particle; and if the velocity (v) of the particle be great, the velocity of the fluid in the vicinity of the particle will also be great. Adopting the customary notation, we have

$$p = \varpi\epsilon^{-\frac{1}{k}\left(\frac{d\phi}{dt} + \frac{1}{2}U^2\right)}.$$

Now, without inquiring into the nature of the function $\frac{d\phi}{dt}$, which for an elastic fluid is no easy matter, all that is necessary is that $\frac{d\phi}{dt} + \frac{1}{2}U^2$ be positive in the vicinity of the particle. In this case p will be diminished; and if we suppose ρ to be unchanged at a considerable distance from the centre, the motion, or the *tendency* to motion, will be symmetrical about the centre, and will be the same as for an incompressible fluid; that is,

$$r^2V = f(t).$$

Differentiating with respect to t,

$$\frac{dV}{dt} = \frac{f'(t)}{r^2} - \frac{2V^2}{r}.$$

Now, initially, $V = 0$; hence the instantaneous effect produced is

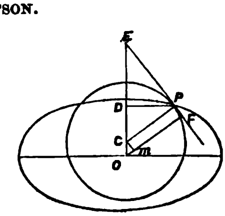
$$\left(\frac{dV}{dt}\right)_{t=0} = \frac{f'(0)}{r^2};$$

which varies inversely as the square of the distance. If $\frac{d\phi}{dt}$ were known in any particular case, as for instance a spherical particle, we could determine $f'(0)$; which will be a function of v. If the motion of the particle be uniform, the acceleration will vary inversely as the square of the distance; but if v be variable, the accelerating force will also be variable. In support of these premises some force may be adduced from the fact that comets seem to contract as they approach the sun, that is, as the velocity increases. Here, although the solid nucleus of the comet may have no sensible disturbing effect on a distant body, yet on the flimsy atmosphere of the comet itself this may be otherwise. On this theory we can account for the tail of a comet. Since the sun is in motion, and the effect produced varies inversely as the square of the distance, the density of the fluid will increase as the distance diminishes. Hence, as the comet approaches the sun, its atmosphere, if of less density than that of the surrounding fluid, will recede from the sun, as smoke and vapour rise in the atmosphere of the earth. As the motion of the earth is not quite uniform, it might be expected that this would have some influence on the moon. There is an annual acceleration in the moon's motion of 2″, which has not been accounted for. May it not be attributed to this cause?

If a solid particle receive a sudden impulse, it will suffer a momentary reaction from the fluid, called inertia; and the equilibrium being restored, and the fluid set in motion, it will continue to flow from fore to aft, and the particle will move on with a uniform motion, without suffering further resistance.

2910. (Proposed by the Rev. J. Wolstenholme, M.A.)—A circle (of radius r) is described with its centre on the minor axis of a given ellipse (of eccentricity e) at a distance from the centre of the ellipse equal to er: prove that the tangents to this circle at any points where it meets the ellipse will touch the minor auxiliary circle.

I. *Solution by* Stephen Watson.



Let O be the centre of the ellipse; C that of the circle; P a point of intersection; EF a tangent to the circle at P meeting OC produced in E, and a perpendicular from O in F; also draw PD perpendicular to OE, and Cm to OF. Put $OD = z$; then $r^2-(z-er)^2 = DP^2 = a^2 - \frac{a^2}{b^2}z^2$,

and this quadratic in z gives $cz + \frac{b^2r}{a} = ab$,

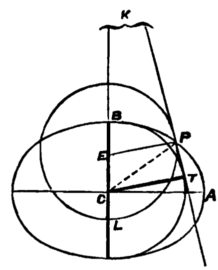
therefore $\qquad e(z-er)+r = e\,.\,\mathrm{CD}+r = b$;

but by similar triangles $\mathrm{CP}:\mathrm{CD} = \mathrm{CO}:\mathrm{O}m$, whence $\mathrm{O}m = e\,.\,\mathrm{CD}$,

therefore $\qquad \mathrm{OF} = \mathrm{O}m + m\mathrm{F}$, or $\mathrm{CP} = e\,.\,\mathrm{CD}+r = b$,

which proves the theorem.

II. *Solution by* R. TUCKER, M.A.



Let E be the centre of the variable circle cutting the ellipse in P, and CT a perpendicular from the centre of the ellipse on the tangent to the circle at P; also let $\angle\,\mathrm{PCT}=\phi$, $\angle\,\mathrm{ACP}=\theta$.

Then $\qquad \dfrac{\sin\phi}{\cos\theta} = \dfrac{\sin \mathrm{CPE}}{\sin \mathrm{PCE}} = \dfrac{\mathrm{CE}}{\mathrm{PE}} = e$;

therefore $\qquad \sin\phi = e\cos\theta$ ....(1);

but $\qquad \mathrm{CT} = \mathrm{CP}\cos\phi$

$$= \frac{b}{\sqrt{(1-e^2\cos^2\theta)}}\,.\,\sqrt{(1-e^2\cos^2\theta)} = b;$$

therefore PT touches the minor auxiliary circle.

2004. (Proposed by W. S. BURNSIDE, M.A.)—Find the condition that the line $\alpha x+\beta y+\gamma z$ may be a normal to the conic

$$(a, b, c, f, g, h)(x, y, z)^2 = 0.$$

I. *Solution by* W. SPOTTISWOODE, M.A., F.R.S.

The condition that the line $lx+my+nz=0$ may touch the conic is, as is well known, $\qquad (\mathrm{A}, \ldots \mathrm{F}, \ldots)(l, m, n)^2 = 0$,

where, as usual, $\mathrm{A} = bc-f^2, \ldots$ The condition that the line $\alpha x+\ldots=0$ may be perpendicular to $lx+\ldots=0$ is

$$(-1,-1,-1,\cos\mathrm{A},\cos\mathrm{B},\cos\mathrm{C})(\alpha,\beta,\gamma)(ax+hy+gz,\; hx+by+fz,\; gx+fy+cz) = 0.$$

And, eliminating x, y, z from this equation and the equations to the line and to the conic, and writing for brevity

$$p=\cos\mathrm{A},\quad q=\cos\mathrm{B},\quad r=\cos\mathrm{C},\quad (-1,-1,-1,p,q,r) = (\,.\,),$$

$$l = (\,.\,)(\alpha,\beta,\gamma)(a,h,g),\quad m = (\,.\,)(\alpha,\beta,\gamma)(h,b,f),\quad n = (\,.\,)(\alpha,\beta,\gamma)(g,f,c),$$

the condition required becomes

$$(a, \ldots f, \ldots)(\beta n-\gamma m,\; \gamma l-\alpha n,\; \alpha m-\beta l)^2 = 0$$

$$= (cm^2-2fmn+bn^2)\alpha^2 + \ldots - 2(fl^2+amn-hnl-glm)\beta\gamma - \ldots.$$

Now $\qquad cm^2-2fmn+bn^2 = \mathrm{A}\left\{(a\ldots)(-1,r,q)^2\alpha^2 + \ldots\ldots\right.$

$$\left.\ldots\ldots + 2(a\ldots)(r,-1,p)(q,p,-1)\beta\gamma + \ldots\right\} - \nabla(\alpha+\beta+\gamma)^2,$$

whence the total expression may be written as follows:—

$$= (\mathrm{A}, \dots \mathrm{F}, \dots)(\alpha, \beta, \gamma)^2 . (a, \dots f, \dots)\left\{(-1, r, q)\alpha, (r, -1, p)\beta, (q, p, -1)\gamma\right\}^2 - \nabla(\alpha+\beta+\gamma)^4 = 0.$$

[Mr. Burnside remarks that "the condition may be put under the form

$$\Delta\Omega^2 + \Sigma(\Phi - \Theta_1\Omega) = 0,$$

where $\Omega = \alpha^2+\beta^2+\gamma^2 - 2\beta\gamma\cos\mathrm{A} - 2\gamma\alpha\cos\mathrm{B} - 2\alpha\beta\cos\mathrm{C}$, $\Delta\Phi = \Theta\Sigma - \Pi^2$, and the remaining notation as in Quest. 1778," (*Reprint*, Vol. V., p. 27).]

II. *Solution by* R. Ball, M.A.; W. H. Laverty, M.A.; *and others.*

Let (x', y', z') be the coordinates of one of the points of intersection of the line with the conic. The tangent to the conic at this point is

$$x(ax'+hy'+gz') + y(hx'+by'+fz') + z(gx'+fy'+cz') = 0;$$

if this be at right angles to $\alpha x + \beta y + \gamma z = 0$,

$$\alpha(ax'+hy'+gz') + \beta(hx'+by'+fz') + \gamma(gx'+fy'+cz') = 0.$$

Write $\alpha^2+\beta^2+\gamma^2 = \rho^2$, $(a, b, c, d, e, f)(\alpha, \beta, \gamma)^2 = \mathrm{S}$,

$$a\alpha + h\beta + g\gamma = \mathrm{L}, \quad h\alpha + b\beta + f\gamma = \mathrm{M}, \quad g\alpha + f\beta + c\gamma = \mathrm{N};$$

then the problem is reduced to the elimination of x, y, z between the equations $\mathrm{L}x + \mathrm{M}y + \mathrm{N}z = 0$, $\alpha x + \beta y + \gamma z = 0$,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0.$$

This is the determinant

$$\begin{vmatrix} \mathrm{L}, & \mathrm{M}, & \mathrm{N}, & 0, & 0 \\ \alpha, & \beta, & \gamma, & 0, & 0 \\ a, & h, & g, & \alpha, & \mathrm{L} \\ h, & b, & f, & \beta, & \mathrm{M} \\ g, & f, & c, & \gamma, & \mathrm{N} \end{vmatrix} = 0,$$

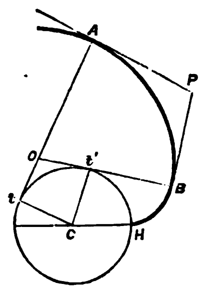
which may be reduced to

$$\begin{vmatrix} 0, & 0, & 0, & -\rho^2, & -2\mathrm{S} \\ \alpha, & \beta, & \gamma, & 0, & -\rho^2 \\ a, & h, & g, & \alpha, & 0 \\ h, & b, & f, & \beta, & 0 \\ g, & f, & c, & \gamma, & 0 \end{vmatrix} = 0, \quad \text{or} \quad 2\mathrm{S}\begin{vmatrix} \alpha, & \beta, & \gamma, & 0 \\ a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \end{vmatrix} + \rho^4 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

2955. (Proposed by M. W. Crofton, F.R.S.)—1. If T, T′ are tangents drawn from any point to the involute of a circle (radius $=1$), and Σ the arc they intercept on the curve, then $\mathrm{T}^2 - \mathrm{T}'^2 = 2(\mathrm{T} + \mathrm{T}' - \Sigma)$.

2. Let AOB, A′OB′ be two concentric quadrants, whose sides coincide. If the points A, B′ be joined by the arc AB′ of an involute of a circle, which arc touches the circular arcs at those points, the arc AB′ is an arithmetical mean between the circular arcs AB, A′B′.

I. *Solution by* William Roberts, Jun.



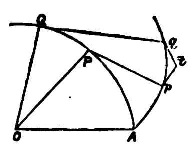
1. Let $AP = T$, $B'P = T'$, $HCt = \theta$, $HCt' = \theta'$;
then (since radius $= 1$) $At = \theta$, $B't' = \theta'$,
$Ot = Ot' = \tan \frac{1}{2}(\theta - \theta')$, $\Sigma = \text{arc } AB' = \frac{1}{2}(\theta^2 - \theta'^2)$,

$$\begin{aligned} T^2 - T'^2 &= OB'^2 - OA^2 = (\theta' + Ot')^2 - (\theta - Ot)^2 \\ &= -\theta^2 + \theta'^2 + 2(\theta + \theta') \tan \tfrac{1}{2}(\theta - \theta') \\ &= 2\left\{(\theta + \theta') \tan \tfrac{1}{2}(\theta - \theta') - \Sigma\right\}. \end{aligned}$$

But
$$\begin{aligned} &(\theta + \theta') \tan \tfrac{1}{2}(\theta - \theta') \\ &= (OA + OB') \tan \tfrac{1}{2}(AOB') = T + T', \end{aligned}$$
therefore $$T^2 - T'^2 = 2(T + T' - \Sigma).$$

2. Suppose AOB' a right angle, then $\theta - \theta' = \frac{1}{2}\pi$,
$$OA = \theta - Ot = \theta - 1, \quad OB' = \theta' + Ot' = \theta' + 1;$$
therefore $$\text{arc } AB = \tfrac{1}{2}(\theta - 1)\pi, \quad \text{arc } A'B' = \tfrac{1}{2}(\theta' + 1)\pi;$$
therefore $$\text{arc } AB + \text{arc } A'B' = \tfrac{1}{2}(\theta + \theta')\pi = 2\Sigma.$$

II. *Solution by the* Rev. J. Wolstenholme, M.A.



1. Let the involute start from A; and let p, q be two points in the involute the tangents at which meet in t; O the centre of the generating circle; P, Q points on the circle corresponding to p, q; $AOP = \theta$, $AOQ = \phi$; O origin of rectangular coordinates, axis of x along OA. Then the equations of tangents at p, q are
$$x \sin\theta - y \cos\theta = a\theta, \quad x \sin\phi - y \cos\phi = a\phi;$$
and if (X, Y) be the coordinates of t, we have
$$X \sin(\phi - \theta) = a(\phi \cos\theta - \theta \cos\phi), \quad Y \sin(\phi - \theta) = a(\phi \sin\theta - \theta \sin\phi);$$
whence, if $pt = T$, $qt = T'$,
$$\left.\begin{aligned} T \sin(\phi - \theta) &= a\left\{\phi - \theta \cos(\phi - \theta) - \sin(\phi - \theta)\right\} \\ T' \sin(\phi - \theta) &= a\left\{\theta - \phi \cos(\phi - \theta) + \sin(\phi - \theta)\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$
Also $2\Sigma = a(\phi^2 - \theta^2)$,
$$\begin{aligned} (T + T') \sin(\phi - \theta) &= a(\phi + \theta)\left\{1 - \cos(\phi - \theta)\right\}, \\ (T^2 - T'^2) \sin^2(\phi - \theta) &= a^2(\phi^2 - \theta^2) \sin^2(\phi - \theta) \\ &\quad - 2a^2(\phi + \theta) \sin(\phi - \theta)\left\{1 - \cos(\phi - \theta)\right\} \\ &= 2a\Sigma \sin^2(\phi - \theta) - 2a(T + T') \sin^2(\phi - \theta), \end{aligned}$$
or $$T'^2 - T^2 = 2a(T + T' - \Sigma).$$

2. Take $\phi - \theta = \frac{1}{2}\pi$, then $T' - T = a(2 - \frac{1}{2}\pi)$, by equations (1);
therefore $2(T + T' - \Sigma) = (1 - \frac{1}{2}\pi)(T + T')$, or $2\Sigma = \frac{1}{2}\pi(T + T')$,
or the arc is an arithmetic mean between quadrants of two circles whose radii are T, T'.

2692. (Proposed by G. M. Smith, B.A.)—Demonstrate by means of a spherical triangle that the same relation between the amplitudes ϕ, ψ, σ, which gives $F(\phi) + F(\psi) - F(\sigma) = 0$ gives also

$$E(\phi) + E(\psi) - E(\sigma) = \kappa^2 \sin\phi \sin\psi \sin\sigma,$$

when κ is the modulus of the elliptic functions.

Solution by the Proposer.

If ϕ, ψ, σ be the sides a, b, c, we have

$$\cos A = \Delta(\phi); \quad \cos B = \Delta(\psi); \quad \cos C = -\Delta(\sigma);$$

$$\kappa = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Now $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

therefore $$\cos A da + \cos B db + \cos C dc = \frac{(\cos a - \cos b \cos c) \sin a da + \dots}{\sin a \sin b \sin c}$$

$$= \frac{d(\cos a \cos b \cos c) - \frac{1}{2} d(\cos^2 a + \cos^2 b + \cos^2 c)}{\sin a \sin b \sin c}$$

$$= \frac{d(2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c)}{2 \sin a \sin b \sin c} \dots\dots (1).$$

But $$\frac{\sin^2 A}{\sin^2 a} = \kappa^2 = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c};$$

therefore (1) may be written

$$\cos A da + \cos B db + \cos C dc = \kappa^2 \frac{d(\sin a \sin b \sin c)^2}{2 \sin a \sin b \sin c} = \kappa^2 d(\sin a \sin b \sin c).$$

Hence, by integrating, and observing that $\int \cos A da = E(\phi)$, &c.,

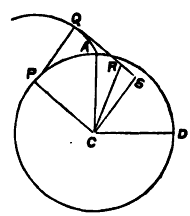
$$E(\phi) + E(\psi) - E(\sigma) = \kappa^2 \sin\phi \sin\psi \sin\sigma.$$

2716. (Proposed by the Rev. James White, M.A.)—If a circular wheel roll against a vertical wall, show that the curve traced on the wheel by a fixed point in the wall is $\theta = \sin^{-1}\frac{d}{\rho} + \frac{(\rho^2 - d^2)^{\frac{1}{2}}}{a}$, where d is the difference between the radius (a) of the circle and the perpendicular height of the fixed point from the ground.

I. *Solution by the* Proposer.

The curve may be obtained by considering the circle fixed, and let the ground line roll round it; thus it may be defined as the locus of a point at a fixed distance on the perpendicular to the normal of the involute, as the ground line becomes that normal. (This gives one half the curve on the circle, the other half being equal and opposite.) In other words, it is the locus of the extremity of a tangent of fixed length to the involute.

Taking the particular case in which that length is the radius of the circle, the curve is the Spiral of Archimedes $\rho = a\omega$; for that is the locus of the foot of the perpendicular from centre of circle on tangent to involute, the part intercepted being manifestly equal to the radius. From this the general equation for any length (p) of the tangent can be easily deduced.



Let PQ be the normal and QS the tangent to the involute of the circle PAD. Then let $QS = CP = a$, and S is a point on the spiral $\rho = a\omega$, and CS is its radius vector. Let $QR = p$, or $RS\ (= a - p) = d$; and it is required to find the equation of the locus of R. CR will be its radius vector.

$CR^2 = CS^2 + SR^2 = a^2\omega^2 + d^2$; *i. e.*, $\rho^2 = a^2\omega^2 + d^2$;

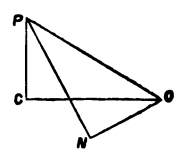
therefore $$\omega = \frac{(\rho^2 - d^2)^{\frac{1}{2}}}{a};$$

but ω is the angle made by CS with CD (at right angles to CA); therefore, in terms of θ, the angle made by CR (ρ) with CD, $\omega = \theta - \sin^{-1}\frac{d}{\rho}$;

therefore $$\theta = \sin^{-1}\frac{d}{\rho} + \frac{(\rho^2 - d^2)^{\frac{1}{2}}}{a}.$$

By a rack of the form of this curve a moving fulcrum can be controlled. Of this Captain Moncrieff has made most valuable and important use in his gun carriage. The sudden and violent strain of the recoil of a heavy gun, before supposed by the best judges to be uncontrollable, has been utilized. The manner in which this curve was practically obtained has caused the rack to be sometimes called cycloidal. But this is a confusion of thought, as the cycloid is generated by a point fixed on the rolling circle, this curve by a point on the wall against which it rolls.

II. *Solution by* C. W. Merrifield, F.R.S.



Let $PC = d$, $PO = r$, $\angle PON = \theta$; also let $\angle CON = \mu$ be the rotation of a diameter, and $CO = a\mu$ the translation of the centre. Then we have at once

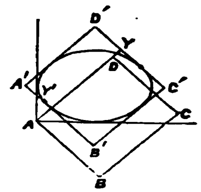
$$r^2 - d^2 = a^2\mu^2,$$

and $$\sin(\theta - \mu) = \sin POC = \frac{d}{r};$$

therefore $$\theta = \sin^{-1}\frac{d}{r} + \frac{(r^2 - d^2)^{\frac{1}{2}}}{a}.$$

2951. (Proposed by M. W. Crofton, F.R.S.)—A given closed curve, whose equation is $\rho = f(\phi)$, moves without rotation so as always to subtend $90°$ at a fixed point A. In any position let ABCD be a circumscribed rectangle; show that the envelope of the sides BC or CD will be given by the equation $$\rho = f(\phi + \tfrac{1}{4}\pi) + f(\phi - \tfrac{1}{4}\pi).$$

Solution by R. W. GENESE.



Let A′B′C′D′ be any circumscribed rectangle; then we get a corresponding position of ABCD by reducing the perpendicular AY ($=p$) on D′C′ by DY ($=$AY′$=p'$), the perpendicular on A′B′. A parallel through D to A′B′ gives DC, &c.

Now if ϕ be the inclination of any tangent to the curve to a fixed line through A, p the perpendicular upon it from A, ρ the radius of curvature at the point of contact, we have the relation

$$\rho = p + \frac{d^2p}{d\phi^2}.$$

The equation to the envelope of DC is therefore

$$\rho = (p-p') + \frac{d^2(p-p')}{d\phi^2}.$$

But if, fixing the initial line properly, $p + \dfrac{d^2p}{d\phi^2} = f(\phi + \frac{1}{4}\pi)$, it is easy to see that

$$p' + \frac{d^2p'}{d\phi^2} = -f(\phi - \tfrac{1}{4}\pi);$$

therefore

$$\rho = f(\phi + \tfrac{1}{4}\pi) + f(\phi - \tfrac{1}{4}\pi).$$

2342. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Construct the maximum triangle of given species whose three sides shall touch three given circles.

Solution by the PROPOSER.

Through the centres of the three circles draw three parallels to the three sides of the triangle. The new will evidently be the maximum with the original triangle; but the sides of the new pass through three given points, and the problem is consequently reduced to the one solved in my treatise on *Modern Geometry*, § 55.

2463. (Proposed by C. M. INGLEBY, LL.D.)—It is required to divide any rational function of n into two parts, such that the one, and the reciprocal of the other, are similar functions of n and of $n+1$ respectively.

I. *Solution by* W. H. LAVERTY, M.A.

We have to find F(n) so that

$$F(n) + \frac{1}{F(n+1)} = f(n),$$

$f(n)$ being a known function. Assume $F(0) = k$. Then

$$F(n+1) = \frac{1}{f(n) - F(n)} = \frac{1}{f(n)}\ \frac{-1}{f(n-1)}\ \frac{-1}{f(n-2)} \cdots\cdots \frac{-1}{f(1)}\ \frac{-1}{f(0)-k}$$

$$= \frac{1}{f(n) - P} \text{ suppose;}$$

and the two parts of $f(n)$ are P and $\{f(n) - P\}$.

II. *Solution by the* PROPOSER.

Let p be the given function, and $f(n)$ and $f(n+1)$ the functions involved. Let us put p under the form $q + \frac{1}{q}$; so that we must have

$$q = \tfrac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}.$$

Then
$$p \equiv \frac{p \pm (p^2-4)^{\frac{1}{2}}}{2} + \frac{2}{p \pm (p^2-4)^{\frac{1}{2}}}.$$

Now, in treating Question 1521, I proved the following theorem, viz.:—

If s be the sum of the odd terms, and s' the sum of the even terms, of a geometrical series whose common ratio is a, and the number of terms be $2n+1$; and σ, σ' be the like when the number of terms are $2(n+1)+1$,

$$\frac{\sigma}{\sigma'} + \frac{s'}{s} = a + \frac{1}{a}.$$

Accordingly, let us put $a = \frac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}$,
then summing both series, we get

$$p = \frac{p \pm (p^2-4)^{\frac{1}{2}}}{2} + \frac{2}{p \pm (p^2-4)^{\frac{1}{2}}}$$

$$= \frac{\left[\frac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}\right]^{2(n+2)} - 1}{\frac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}\left\{\left[\frac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}\right]^{n+1} - 1\right\}}$$

$$+ \frac{\frac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}\left\{\left[\frac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}\right]^{2n} - 1\right\}}{\left[\frac{1}{2}\left\{p \pm (p^2-4)^{\frac{1}{2}}\right\}\right]^{2(n+1)} - 1}$$

or
$$p = f(n+1) + \frac{1}{f(n)}.$$

2844. (Proposed by A. MARTIN.)—To find three positive integral numbers, whose sum, and also the sum of any two of them, shall be a rational cube.

Solution by S. BILLS.

Let $\tfrac{1}{2}(-x^3+p^3+q^3),\ \ \tfrac{1}{2}(x^3-p^3+q^3),\ \ \tfrac{1}{2}(x^3+p^3-q^3)$ ....(A)

represent the three required numbers; then three of the conditions will be satisfied, and it will only remain to make

$$\tfrac{1}{2}(x^3+p^3+q^3) = \text{a cube} = r^3, \quad \text{or} \quad x^3+p^3+q^3 = 2r^3 \quad \text{.........(1)}.$$

In (1) assume $p = z+1$, $q = z-1$, and $r = z+v$, then we have

$$x^3+2z^3+6z = 2z^3+6z^2v+6zv^2+2v^3,$$

or,
$$z^2+\frac{v^2-1}{v}z = \frac{x^3-2v^3}{6v} \quad \text{.........................(2)}.$$

Solving (2) we find $\quad z = \dfrac{1-v^2}{2v} \pm \dfrac{1}{6v}\sqrt{(6vx^3-3v^4-18v^2+9)}.$

We must, therefore, have $\quad 6vx^3-3v^4-18v^2+9 = \square$. Assume

$$6vx^3-3v^4-18v^2+9 = 6v(x+2a)(x-a)^2.$$

Developing this and reducing, we find

$$x = \frac{v^4+6v^2+4a^3v-3}{6a^2v}, \quad \text{and} \quad 6v(x+2a) = \frac{1}{a^2}(v^4+6v^2+16a^3v-3),$$

which must be a square. Assume

$$v^4+6v^2+16a^3v-3 = (v^2+3)^2 = v^4+6v^2+9,$$

whence $v = \dfrac{3}{4a^3}$; where a may be taken any number that will make *all* the expressions in (A) positive. Assume $a = \tfrac{1}{2}$, then $v = 6$; and thence $x = 168$, $z = 360$, $p = 361$, $q = 359$; and the three numbers will be 44286264, 1982015, and 2759617. These are the least I have been able to find; others may be found at pleasure.

[After two long but elegant investigations, which we have not room enough to publish, the Proposer gives the two following sets of values, as the least he has been able to find satisfying the conditions of the problem:—

5745716741366	1882652672794926592
395664832588085	743505869778978033
105095301929290	281491575688125711.

The first of these was given on p. 118 of Brickley's *Algebra*, published in Dublin in 1811.]

2928. (Proposed by A. MARTIN.)—Prove that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n} = n-\frac{n(n-1)}{1\,.\,2^2}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3^2}-\&c.$$

Solution by the Rev. J. BLISSARD.

1. The sum of the reciprocals, viz. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, can be equated to an indefinite number of functions of n. (See *Quarterly Journal of Mathematics*, Vol. VI., p. 242.) Thus

$$1+\frac{1}{2}+\ldots+\frac{1}{n} = n-\frac{n(n-1)}{1.2^2}+\frac{n(n-1)(n-2)}{1.2.3^2}-\&c. \text{ (as proposed above)}$$

$$= 1+\frac{n-1}{1^2.2}-\frac{(n-1)(n-2)}{1.2^2.3}+\frac{(n-1)(n-2)(n-3)}{1.2.3^2.4}-\&c.$$

$$= \frac{5n-1}{2(n+1)}+\frac{2}{n+1}\left\{\frac{(n-1)(n-2)}{1^2.2^2.3}-\frac{(n-1)(n-2)(n-3)}{1.2^2.3^2.4}+\&c.\right\}$$

&c. &c. &c.

$$= \frac{1}{n+1}\left\{2n+\frac{1}{1.2}\cdot\frac{n(n-1)}{1.2}-\frac{1}{2.3}\cdot\frac{n(n-1)(n-2)}{1.2.3}+\&c.\right\}$$

$$= (n-1)+(-1)^n\left\{\frac{n(n-1)}{1.2^2}-2^{n-1}\cdot\frac{n(n-1)(n-2)}{1.2.3^2}+\&c.\right\}$$

&c. &c. &c.

2. The above Question can also be variously generalized. Thus

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+n} = \frac{n}{m+1}-\frac{n(n-1)}{(m+1)(m+2)}\cdot\frac{1}{2} + \frac{n(n-1)(n-2)}{(m+1)(m+2)(m+3)}\cdot\frac{1}{3}-\&c.$$

and

$$= \frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma n}\left\{\frac{1}{(m+1)^2}-\frac{n-1}{1}\cdot\frac{1}{(m+2)^2} + \frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{(m+3)^2}-\&c.\right\}.$$

I propose to prove the following generalization, viz. that

$$\frac{x}{1}+\frac{x^2}{2}+\ldots+\frac{x^n}{n} = \frac{n}{1^2}\left\{1-(1-x)\right\}-\frac{n(n-1)}{1.2^2}\left\{1-(1-x)^2\right\}+\&c. \ldots \text{(I)}.$$

Assume

$$\frac{x}{1}+\frac{x^2}{2}+\ldots+\frac{x^n}{n} = \frac{n}{1}\cdot\frac{u_1}{1}-\frac{n(n-1)}{1.2}\cdot\frac{u_2}{2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{u_3}{3}-\&c. \ldots (a);$$

therefore $\dfrac{x}{1}+\dfrac{x^2}{2}+\ldots+\dfrac{x^{n+1}}{n+1} = \dfrac{n+1}{1}\cdot\dfrac{u_1}{1}-\dfrac{(n+1)n}{1.2}\cdot\dfrac{u_2}{2}+\&c.;$

therefore, subtracting and multiplying by $n+1$,

$$\frac{n+1}{1}\cdot u_1-\frac{(n+1)n}{1.2}\cdot u_2+\&c. = x^{n+1};$$

i. e., putting n for $n+1$, and using Representative Notation,

$$1-(1-u)^n = x^n.$$

Hence, applying Taylor's theorem,

$$f(h+\theta)-f\left\{h+(1-u)\theta\right\} = f(h+x\theta)-fh.$$

Now let $h=1$, $\theta=-1$, and $fz=z^r$;

therefore $\quad -u^r = (1-x)^r-1, \quad$ *i. e.*, $\quad u_r = 1-(1-x)^r.$

Making this substitution in (a), we obtain the formula (I).

COR. 1.—In (I), put $1-x$ for x, then

$$\frac{1-x}{1}+\frac{(1-x)^2}{2}+\ldots+\frac{(1-x)^n}{n}=\frac{n}{1^2}(1-x)-\frac{n(n-1)}{1.2^2}(1-x^2)+\&c.$$

Hence, equating coefficients of x^r, we have

$$1+r+\frac{r(r+1)}{1.2}+\ldots+\frac{r(r+1)\ldots(n-1)}{1.2\ldots(n-r)}=\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\quad(n>r).$$

COR. 2.—In (I), put $-x$ for x, then

$$\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\pm\frac{x^n}{n}=\frac{n}{1^2}\left\{(1+x)-1\right\}-\frac{n(n-1)}{1.2^2}\left\{(1+x)^2-1\right\}+\&c.$$

3. We can now readily prove that

$$\frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}=\frac{n}{1^2}\left\{1-(1-x)\right\}-\frac{n(n-1)}{1.2^2}\left\{1+\frac{1}{2}-(1-x)-\frac{1}{2}(1-x)^2\right\}$$

$$+\frac{n(n-1)(n-2)}{1.2.3^2}\left\{1+\frac{1}{2}+\frac{1}{3}-(1-x)-\frac{1}{2}(1-x)^2-\frac{1}{3}(1-x)^3\right\}-\&c.\ \ldots\ (\text{II}).$$

Let R be the representative of the reciprocals $\frac{1}{1},\frac{1}{2},\ldots\frac{1}{n}$, so that $R^n=\frac{1}{n}$; then, putting Rx for x in (I),

$$\frac{Rx}{1}+\frac{R^2x^2}{2}+\ldots+\frac{R^nx^n}{n},\quad i.e.,\quad \frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}$$

$$=\frac{n}{1^2}\left\{1-(1-Rx)\right\}-\frac{n(n-1)}{1.2^2}\left\{1-(1-Rx)^2\right\}+\ldots\ldots$$

$$\ldots\ldots\pm\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\left\{1-(1-Rx)^2\right\}.$$

But $1-(1-Rx)^r=\frac{r}{1}Rx-\frac{r(r-1)}{1.2}R^2x^2+\&c.=\frac{r}{1^2}x-\frac{r(r-1)}{1.2^2}x^2+\&c.,$

which, from Art. 2, Cor. 1,

$$=\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r}-\left(\frac{1-x}{1}+\frac{(1-x)^2}{2}+\ldots+\frac{(1-x)^r}{r}\right).$$

Hence, making this substitution, we obtain Formula (II).

COR. 1.—Put $1-x$ for x in (II), then

$$\frac{1-x}{1^2}+\frac{(1-x)^2}{2^2}+\ldots+\frac{(1-x)^n}{n^2}=\frac{n}{1^2}(1-x)-\frac{n(n-1)}{1.2^2}\left\{1+\frac{1}{2}-\left(x+\frac{x^2}{2}\right)\right\}+\ldots$$

$$\ldots+(-1)^{r+1}\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\left\{1+\frac{1}{2}+\ldots+\frac{1}{r}-\left(x+\ldots+\frac{x^r}{r}\right)\right\}+\&c.$$

Hence, equating coefficients of x^r, we have $(n>r)$

$$\frac{1}{r}+\frac{r}{1}\cdot\frac{1}{r+1}+\frac{r(r+1)}{1.2}\cdot\frac{1}{r+2}+\ldots+\frac{r(r+1)\ldots(n-1)}{1.2\ldots(n-r)}\cdot\frac{1}{n}$$

$$=(-1)^{n-r}\left\{\frac{1}{n}-\frac{n}{1}\cdot\frac{1}{n-1}+\frac{n(n-1)}{1.2}\cdot\frac{1}{n-2}-\ldots\pm\frac{n(n-1)\ldots(r+1)}{1.2\ldots(n-r)}\cdot\frac{1}{r}\right\}.$$

Cor. 2.—In (II), put $x=1$, then

$$\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}=\frac{n}{1^2}.\Sigma.1-\frac{n(n-1)}{1.2^2}.\Sigma.\frac{1}{2}+\frac{n(n-1)(n-2)}{1.2.3^2}.\Sigma.\frac{1}{3}-\dots$$

where $$\Sigma\frac{1}{n}=1+\frac{1}{2}+\dots+\frac{1}{n}.$$

II. *Solution by* W. H. Laverty, B.A.; Rev. J. Wolstenholme, M.A.; *the* Proposer; *and many others.*

We have $$(1-1)^n=0=1-n+\frac{n(n-1)}{1.2}-\dots\mp n\pm 1,$$

(where the upper signs refer to n as even, and the lower to n as odd,)

therefore $$\frac{1\pm1}{n}=1-\frac{(n-1)}{1.2}+\frac{(n-1)(n-2)}{1.2.3}-\dots\pm1.$$

Now assume the equality true for $(n-1)$; that is, that

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-1}=(n-1)-\frac{(n-1)(n-2)}{1.2^2}+\dots$$
$$\dots\pm\frac{(n-1)\dots(n-k)}{1.2\dots k^2}\mp\dots\pm\frac{(n-1)\dots1}{1.2\dots(n-1)^2};$$

also

$$\frac{1+1}{n}=1-\frac{(n-1)}{1.2}+\dots\pm\frac{(n-1)\dots(n-k+1)}{1.2\dots k}\mp\dots\pm\frac{(n-1)\dots2}{1.2\dots(n-1)};$$

therefore by addition we obtain

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-1}+\frac{1\pm1}{n}=n-\frac{n(n-1)}{1.2^2}+\dots$$
$$\dots\pm\frac{n(n-1)\dots(n-k+1)}{1.2.3\dots k^2}\mp\dots\pm\frac{n(n-1)\dots2}{1.2\dots(n-1)^2},$$

or $$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-1}+\frac{1}{n}=n-\frac{n(n-1)}{1.2^2}+\dots\mp\frac{n}{n-1}\pm\frac{1}{n}.$$

If, therefore, the equality be true for $(n-1)$, it is true for n; but it is true for 2, 3, &c., therefore it is true generally.

2977. (Proposed by Professor Sylvester.)—Let x be any positive integer, and $1.3.5\dots(2x-1)=P_x$. Required to prove that

$$xP_{x-1}+\frac{x(x-1)}{2}2P_{x-2}+\frac{x(x-1)(x-2)}{3}2^2P_{x-3}+\dots+\frac{x(x-1)\dots1}{x}2^{x-1}$$
$$=P_x\left(1+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2x-1}\right).$$

Solution by J. J. WALKER, M.A.

The left-hand side of the equality to be proved may be written

$$2^{x-1}.1.2.3\ldots x\left(\frac{1.3.5\ldots 2x-3}{1.2.3\ldots x-1}\frac{1}{2_{x-1}}+\frac{1}{2}\frac{1.3.5\ldots 2x-5}{1.2.3\ldots x-2}\frac{1}{2^{x-2}}\right.$$
$$\left.+\frac{1}{3}\frac{1.3.5\ldots 2x-7}{1.2.3\ldots x-3}\frac{1}{2^{x-3}}+\ldots+\frac{1}{x-1}\frac{1.3}{1.2}\frac{1}{2}+\frac{1}{x}\right)\ldots\ldots\ldots(1).$$

The series within brackets is easily seen to be identical with the coefficient of z in the product of the expansions of $(-1)^{x-1}(1+z)^{-\frac{1}{2}}$ and of $\log_e(1+z)$, which coefficient is otherwise equal to $\dfrac{(-1)^{x-1}}{1.2\ldots x}\dfrac{d^x.y^{-\frac{1}{2}}\log_e y}{dy^x}$, when $y=1$. Now it will readily be found, by induction from the first four or five of the successive differential coefficients of $y^{-\frac{1}{2}}\log_e y$, that

$$\frac{d^x.y^{-\frac{1}{2}}\log_e y}{dy^x}=(-1)^{x-1}\frac{P_x}{2^{x-1}}y^{-\frac{2x+1}{2}}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2x-1}-\frac{\log_e y}{2}\right),$$

which, multiplied by $\dfrac{(-1)^{x-1}}{1.2\ldots x}$ and by the factors outside brackets in (1), becomes identically equal to the right-hand side of the equality to be proved, when $y=1$.

[As examples, give x the values 3 and 4; then

$$3(1.3)+3.2(1)+2.2^2=23=1.3.5+1.5+1.3;$$
$$4(1.3.5)+6.2(1.3)+8.2^2(1)+6.2^3=176=3.5.7+5.7+3.7+3.5.$$

Prof. SYLVESTER remarks that this theorem occurred to him as a corollary to a theorem in his theory of Reducible Cyclodes.]

NOTE ON GRAVITATION. *By* G. O. HANLON.

Some years ago a theory of gravitation occurred to me which has been brought to my mind by Mr. Tebay's paper on Gravitation in the last number of the *Educational Times*. I would first remark, however, on what I consider to be a very weak point in Mr. Tebay's theory, namely, that, under his system, the force of attraction of a body would vary as its volume and not as its mass. The only way in which the mass, independently of the volume, would, in his theory, seem to be able to exercise any effect on other bodies, would be by the power of the body in motion to remove more or less ether from its path in a given time. This, no doubt, depends on its mass; but the hypothesis becomes untenable when we reflect that the sun and planets, all of different densities, have the same velocity round their common centre.

Some philosophers, in order to account for the constant replenishing of the solar heat, maintain that the sun is constantly receiving showers of meteors on its surface. It has been calculated that the quantity of these

bodies which the sun must receive, in order to have his supply of heat maintained, should be sufficient to cover that body all round to the depth of sixty feet in one year. But what is the likelihood of such a supply of heat coming from meteors? Comets, which are of such slight specific gravity, pass very close to the sun (that of 1680 passed within a sixth of his diameter), yet in the memory of man not one has been seen to be absorbed in its mass. I find that, if the meteors which we observe in our own atmosphere are part of a system of meteors moving from all points of the heavens towards the sun, (and it is likely they only occupy that portion of space near the plane of the ecliptic,) that luminary must receive 515 million times the quantity which falls on the earth. Yet we know that very few fall on the earth, while there is every reason to believe that the quantity we meet with every cycle of 33 years is a ring of such meteors moving round, and not into, the sun. This theory does not, therefore, account satisfactorily for the replenishing of the solar heat.

The diminution of the periods of Encke's comet would appear to establish the existence of the *material* ether Mr. Tebay speaks of; and the question naturally arises, would not a motion of the ether towards the sun account for the constant supply of heat, and furnish an explanation of the phenomenon of gravitation. If we suppose every particle of matter in the universe to be constantly absorbing or, as it were, feeding on the surrounding ether, we shall have a motion of that fluid towards the sun and each planet which would account for all the phenomena of gravitation. The force of the ether moving each planet towards the sun would vary inversely as the square of the distance, while towards each of the planets there would be a minor flow of the ether, proportionate to its mass, which would keep the satellites in their orbits. Thus the fundamental requirements of a theory of gravitation would be complied with, the force towards a body varying as the mass, and inversely as the square of the distance, while we are led also towards the solution of the much discussed question of the counteracting influence to the radiation of the sun's heat.

I would further observe that Mr. Tebay draws a conclusion that the nearer to the body the denser the ether. I cannot avoid drawing an opposite conclusion, both in his supposition and mine (except for the part immediately before the moving body). For a long time I endeavoured to prove it would be as he says, since the diurnal motion of the earth, planets, and sun would then be accounted for by the inertia of the ether on the side of a body next the centre of force being greater than that on the opposite side.

2996. (Proposed by J. J. WALKER, M.A.)—If from the three angles of a triangle, which is circumscribed by a conic, lines be drawn parallel to those joining the middle points of the sides opposite those angles respectively with the centre of the conic, the three lines so drawn will cointersect.

Solution by S. WATSON; REV. T. J. SANDERSON, M.A.; F. D. THOMSON, M.A.; R. W. GENESE; *and others.*

This property is true, if, instead of the centre of the conic, we take any point in the plane of the triangle. Let any such point be determined by

$$l\alpha = m\beta = n\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

then the equations of lines joining this point to the middle points of the sides are $l(cm-bn)\alpha+mn(b\beta-c\gamma)=0$, $m(an-cl)\beta+nl(c\gamma-a\alpha)=0$,

$$n(bl-am)\gamma+lm(a\alpha-b\beta)=0,$$

and if $\quad p=\dfrac{l(bn-cm)}{a},\quad p_1=\dfrac{m(cl-an)}{b},\quad p_2=\dfrac{n(am-bl)}{c},$

the equations of lines through the angles parallel to the preceding are respectively

$$\left.\begin{aligned}(mn+p)\,b\beta &= (mn-p)\,c\gamma\\ (nl+p_1)\,c\gamma &= (nl-p_1)\,a\alpha\\ (lm+p_2)\,a\alpha &= (lm-p_2)\,b\beta\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

When these meet in a point, we have

$$(mn+p)(nl+p_1)(lm+p_2)=(mn-p)(nl-p_1)(lm-p_2),$$

which reduces to $\quad lmn(lp+mp_1+np_2)+pp_1p_2=0;$

and substituting the above values of p, p_1, p_2, the result vanishes identically; hence the lines (2) co-intersect.

[The theorem in the Question is the projection of the theorem that the perpendiculars of a triangle meet in a point, as may be seen by making the circumscribing conic a circle, and then projecting orthogonally the circle into the conic.]

THE "TRUE REMAINDER." *By* ARTEMAS MARTIN.

Let it be required to divide D by d, d being a composite number.

Put $d=d_1d_2d_3d_4\ldots d_n$; $q_1, q_2, q_3, q_4, \ldots q_n$ = the several quotients respectively; $r_1, r_2, r_3, r_4, \ldots r_n$ = the respective remainders left by the divisors $d_1, d_2, d_3, d_4, \ldots d_n$; and $R_1, R_2, R_3, R_4, \ldots R_n$ = the *true* remainders after each division.

The operation will stand thus:—

$$\begin{array}{rlll}
d_1) & \underline{D} & & \\
d_2) & \underline{q_1} & \ldots\ r_1, & \text{first remainder;}\\
d_3) & \underline{q_2} & \ldots\ r_2, & \text{second}\quad ,,\\
d_4) & \underline{q_3} & \ldots\ r_3, & \text{third}\quad ,,\\
 & q_4 & \ldots\ r_4, & \text{fourth}\quad ,,\\
 & \vdots & \ \ \vdots & \ \ \vdots\\
d_n) & \underline{q_{n-1}} & & \\
 & q_n & \ldots\ r_n, & \text{last}\quad ,,
\end{array}$$

Reversing the above operation, we have

$$d_1q_1+r_1=D,\ \text{therefore}\ R_1=r_1;$$

$d_2q_2+r_2=q_1$, therefore $d_1q_1=d_1d_2q_2+d_1r_2$, and $d_1d_2q_2+d_1r_2+R_1=D$.

But $\quad D=d_1d_2q_2+R_2$, therefore $R_2=R_1+d_1r_2=r_1+d_1r_2$;

also, $\quad d_3q_3+r_3=q_2$, therefore $d_1d_2q_2=d_1d_2d_3q_3+d_1d_2r_3$,

and $\quad d_1d_2d_3q_3+d_1d_2r_3+R_2=D.$

But $D = d_1 d_2 d_3 q_3 + R_3$; therefore $R_3 = R_2 + d_1 d_2 r_3 = r_1 + d_1 r_2 + d_1 d_2 r_3$.
Similarly, we find

$$R_4 = R_3 + d_1 d_2 d_3 r_4 = r_1 + d_1 r_2 + d_1 d_2 r_3 + d_1 d_2 d_3 r_4;$$

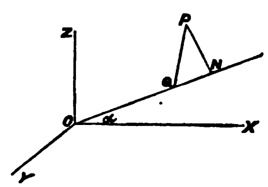
and, *generally*,

$$R_n = R_{n-1} + d_1 d_2 d_3 \dots d_{n-1} r_n$$
$$= r_1 + d_1 r_2 + d_1 d_2 r_3 + d_1 d_2 d_3 r_4 + \dots + d_1 d_2 d_3 \dots d_{n-1} r_n,$$

which is the "Rule" given in Arithmetics.

2952. (Proposed by J. J. WALKER, M.A.)—Find the locus of points from which if a heavy inelastic globule be dropped on a smooth inclined plane, its velocity when passing a given horizontal line drawn on the plane shall be constant.

Solution by R. W. GENESE.



If the particle P fall down PQ, striking the inclined plane at Q, the effect of the impulse will be, that the velocity perpendicular to the plane will be destroyed, and that along the plane be unaffected. But if PN be perpendicular to the plane, by the principle of independent motions,

the velocity along plane acquired in falling down PQ = velocity which would be acquired in falling along NQ. Thus the velocity on passing a fixed horizontal line Oy is that which would be acquired in falling from N to O. This is to be constant; therefore N lies on a fixed straight line. Hence the locus of P is a plane perpendicular to given plane.

2909. (Proposed by S. ROBERTS, M.A.)—In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle in which four foci, real or imaginary, lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle.

Solution by the PROPOSER.

The equation of a quartic with two double points may be written

$$(\alpha\gamma_1 - \alpha_1\gamma)^2 (\beta\gamma_1 - \beta_1\gamma)^2 + (m\alpha + n\beta)(\alpha\gamma_1 - \alpha_1\gamma)(\beta\gamma_1 - \beta_1\gamma)\gamma + (a\alpha + b\beta)^2\gamma^2 = 0,$$

the double points being $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$. The first polar of $\widehat{\alpha\beta}$ is

$$-2\alpha_1(\alpha\gamma_1 - \alpha_1\gamma)(\beta\gamma_1 - \beta_1\gamma)^2 - 2\beta_1(\beta\gamma_1 - \beta_1\gamma)(\alpha\gamma_1 - \alpha_1\gamma)^2$$
$$+ (m\alpha + n\beta)(-\alpha\beta_1\gamma_1 - \beta\alpha_1\gamma_1 + 2\alpha_1\beta_1\gamma_1)\gamma + 2\gamma(a\alpha + b\beta)^2$$
$$+ (m\alpha + n\beta)(\alpha\gamma_1 - \alpha_1\gamma) = 0.$$

Multiply this by 2γ, and subtract from the equation of the curve. Then

$$2(\alpha\gamma_1-\alpha_1\gamma)(\beta\gamma_1-\beta_1\gamma)\left\{(\alpha\gamma_1-\alpha_1\gamma)(\beta\gamma_1-\beta_1\gamma)+\alpha_1\gamma(\beta\gamma_1-\beta_1\gamma)\right.$$
$$\left.+\beta_1\gamma(\alpha\gamma_1-\alpha_1\gamma)\right\}$$
$$+(m\alpha+n\beta)(\alpha\beta\gamma_1^2-\alpha_1\beta_1\gamma^2)=0,$$

or $$\left\{2(\alpha\gamma_1-\alpha_1\gamma)(\beta\gamma_1-\beta_1\gamma)+(m\alpha+n\beta)\gamma\right\}(\alpha\beta\gamma_1^2-\alpha_1\beta_1\gamma^2)=0.$$

But if we change $\alpha_1\beta_1\gamma_1$ into $\alpha_2\beta_2\gamma_2$, the curve represented remaining the same, we have $\alpha_1\beta_1\gamma_2^2-\alpha_2\beta_2\gamma_1^2=0$, that is to say, $\alpha\beta\gamma_1^2-\alpha_1\beta_1\gamma^2=0$ is a conic through $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$, on which four intersections of tangents from $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$ lie. Hence it follows, &c. The other factor determines the points of contact of the double tangents; these also consequently lie in another conic through $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$.

Let now $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$ represent the circular points at infinity; γ is the line at infinity, and we have a bicircular quartic; the conics become circles, and $(\alpha_1\beta_1\gamma_1)$ represents a focus.

2749. (Proposed by S. WATSON.)—Let D, E, F be the points of contact of the inscribed conic with the triangle of reference, and let AD cut the conic in P_1. Join BP_1, CP_1, meeting the opposite sides in E_1, F_1, and let now D, E_1, F_1 be the points of contact of a second inscribed conic, cutting AD in P_2. Continue this process; then will the equations of the tangent to the nth conic at P_n and of the line $E_n F_n$ be, respectively,

$$l\alpha-\tfrac{1}{2}(4)^n(m\beta+n\gamma)=0, \quad \text{and} \quad l\alpha-(4)^n(m\beta+n\gamma)=0.$$

Solution by the PROPOSER.

Let the equation of the inscribed conic be

$$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0 \quad\ldots\ldots\ldots\ldots(1);$$

then the equation of AD is $m\beta-n\gamma=0$, and eliminating first β and then γ from this and (1), the results

$$l\alpha-4n\gamma=0 \ \text{ and } \ l\alpha-4m\beta=0 \quad\ldots\ldots\ldots\ldots(2),$$

are the equations of BE_1, CF_1; hence the equation of E_1F_1 is

$$l\alpha-4(m\beta+n\gamma)=0 \quad\ldots\ldots\ldots\ldots(3),$$

and that of the conic touching the sides at D, E_1, F_1 is

$$(l\alpha)^{\frac{1}{2}}+(4m\beta)^{\frac{1}{2}}+(4n\gamma)^{\frac{1}{2}}=0 \quad\ldots\ldots\ldots\ldots(4).$$

Again, the equation of any line through D is $m\beta-n\gamma-r\alpha=0$; and if this meet (1) again in P′, the equation of CP′ is $(l+r)^2\alpha-4lm\beta=0$, hence the equation of P_1P' is

$$l^2\alpha-(2l-r)m\beta-(2l+r)n\gamma=0 \quad\ldots\ldots\ldots\ldots(5),$$

and when $r=0$ this becomes a tangent at P_1, whose equation therefore is

$$l\alpha-2(m\beta+n\gamma)=0 \quad\ldots\ldots\ldots\ldots(6).$$

Proceeding in the same manner with (4) as above with (1), we find the equations of E_2F_2 and the tangent at P_2 to be

$$l\alpha-(4)^2(m\beta+n\gamma) = 0 \quad\text{and}\quad l\alpha-\tfrac{1}{2}(4)^2(m\beta+n\gamma) = 0;$$

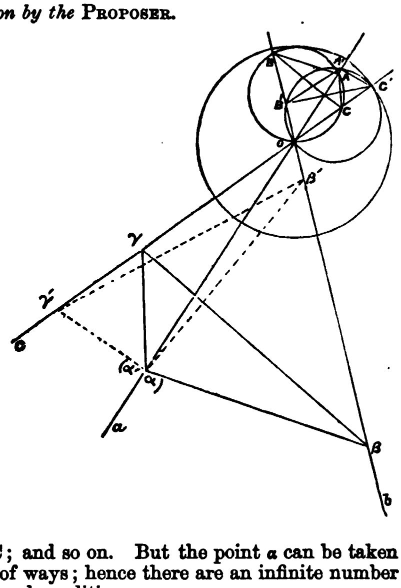
hence, generally, the equations of $E_n\,F_n$ and the tangent at P_n are

$$l\alpha-(4)^n(m\beta+n\gamma) = 0 \quad\text{and}\quad l\alpha-\tfrac{1}{2}(4)^n(m\beta-n\gamma) = 0.$$

COR.—The lines $E_1F_1, \ldots E_n\,F_n$, and the tangents at $P_1, P_2, \ldots P_n$, are all concurrent, as they all pass through the point where the line $m\beta+n\gamma = 0$ meets the side BC.

2410. (Proposed by PAUL BRANQUART.)—If a polygon A, B, C, D, ... M be inscribed in a circle, which is supposed to roll on the inside of a circle of double its radius, prove that from the centre o of the greater circle a pencil of m lines oa, ob, oc, od, ... om can be drawn such that, for any determined position taken by the given polygon, an infinite number of polygons can be drawn homothetical to it, and having their summits $\alpha, \beta, \gamma, \delta, \ldots \mu$ upon the lines oa, ob, oc, od, ... om, respectively.

Solution by the PROPOSER.



Assuming LAHIRE's theorem (see DR. SONNET's *Notions de Mécanique*, p. 10), we remark that, in the revolution of the smaller circle, each summit of the given polygon remains upon a diameter of the greater circle. Consequently, if we consider the pencil formed by the diameters oa, ob, oc, od, ... om, upon which the summits A, B, C, D, ... M of the given polygon move, it is clear that, *to construct a polygon homethetical to the given one, considered in any determined one of the positions it may assume*, it will be sufficient to take on the diameter oa (considered as an infinite line) any point α, and through α draw $\alpha\beta$ parallel to AB; then through β (β being the intersection of $\alpha\beta$ and ob) draw $\beta\gamma$ parallel to BC; and so on. But the point α can be taken upon oa in an infinite number of ways; hence there are an infinite number of polygons fulfilling the proposed conditions.

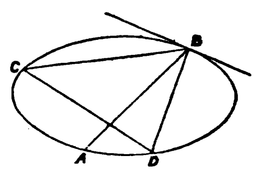
[Mr. BRANQUART explains his figure thus:—In the above diagram, I have endeavoured to illustrate the construction I have described, and have

considered two different positions of the given polygon (here a triangle, to make the figure plainer), and have with the point α constructed a triangle $\alpha\beta\gamma$ homothetical to ABC, and another $\alpha'\beta'\gamma'$ homothetical to A′B′C′.

Lahire's theorem, referred to above, is thus expressed,—"Si un cercle roule sans glisser sur une circonférence de rayon double et intérieurement à cette circonférence, chacun des points de la petite circonférence se meut sur un diamètre de la grande."]

2803. (Proposed by Professor Hirst.)—Through every point A on a conic pass three circles which osculate the conic elsewhere, say in B, C, D. Prove that A, B, C, D lie on the circumference of a circle, and find the envelope of the latter.

Solution by the Rev. J. Wolstenholme, M.A.



Let BCD be a maximum triangle in an ellipse, and let the circle about BCD meet the ellipse again in A; then BA, CD are equally inclined to axis, therefore BA and tangent at B are equally inclined to axis, therefore circle of curvature at B passes through A; similarly circles of curvature at C, D pass through A. (This proof of this well known proposition was given me by Mr. B. W. Horne, of St. John's, who believed it to have been invented by one of his pupils).

If θ be eccentric angle of A, the equation of the circle ABCD may be found immediately to be

$$2(x^2+y^2)-(a^2-b^2)\left(\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta\right)=a^2+b^2,$$

and the envelope is therefore the bicircular quartic

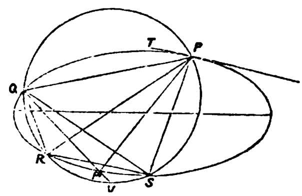
$$\left\{2(x^2+y^2)-(a^2+b^2)\right\}^2=(a^2-b^2)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right).$$

Hence in any such circle the rectangle of segments of chords drawn through the centre of the ellipse is $\frac{1}{2}(a^2+b^2)$, and the envelope of the polar of the centre of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=4\left(\frac{a^2+b^2}{a^2-b^2}\right)^2.$$

2446. (Proposed by W. K. Clifford, B.A.)—PQ is a chord of a conic, equally inclined to the axis with the tangent at P. Any circle through PQ cuts the conic in RS. Show that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P. Hence show by inversion that if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point.

Solution by J. J. WALKER, M.A.



1. Let T be a point on the tangent to the conic at P; PP′ the diameter of the conic through P, and let QP′ meet the circle in V.

Since RS and PQ are equally inclined to the axis, RS is parallel to PT and is bisected by PP′; and the pencil P (TRP′S) is therefore harmonic.

Again, the angle PQP′ = TPP′, because QP′ is parallel to the diameter bisecting PQ, which is equally inclined to the axis as the chords bisected by PP′. Also QS and PR, being equally inclined to the axis, must make equal angles with QP′ and PP′ respectively. For a similar reason the angle P′QR is equal to P′PS. Hence the pencil Q (PSVR) is similar to P (TRP′S), and V is therefore the harmonic to P relatively to R and S.

2. Suppose now P to be the double point on the circular cubic. The inverse of this to P as origin is well known to be a conic. To the asymptote of the cubic will correspond the osculating circle to the conic at P, which cuts the conic in Q, this point being the inverse of that in which the asymptote cuts the cubic. To a chord of the cubic drawn through the latter point will correspond a circle PSVRQ; and to the middle point of the chord, the point (V) in which the harmonic to PR and PS, relatively to the tangent of the circle at P, meets that circle. The locus of V being the fixed line QP′, that of the middle points of the chords must be a circle passing through P.

2946. (Proposed by I. H. TURRELL.)—To find three whole numbers in arithmetical progression whose common difference shall be a cube, the sum of any two diminished by the third a square, and the sum of the roots of the required squares a square.

I. *Solution by* SAMUEL BILLS.

Let x^2-y^3, x^2, x^2+y^3 be the three required numbers; then by the question we must have x^2-2y^3 and x^2+2y^3 both squares. Assume $x=uy$, then we shall have to find u^2-2y and u^2+2y both squares; these conditions, as is well known, will be fulfilled by taking $u = t(p^2+q^2)$ and $y = 2pqt^2(p^2-q^2)$. These values will give

$$u^2-2y = t^2\left\{2pq-(p^2-q^2)\right\}^2 \quad \text{and} \quad u^2+2y = t^2\left\{2pq+(p^2-q^2)\right\}^2;$$

and it will now remain to find $2t(p^2+4pq+q^2)\,pq\,(p^2-q^2) = \square$.

Assume the above $= 4p^2q^2(p^2+4pq+q^2)^2(p^2-q^2)^2r^2$;

then $\quad t = 2pq(p^2-q^2)(p^2+4pq+q^2)r^2$;

whence $\quad u = 2pq(p^2-q^2)(p^2+q^2)(p^2+4pq+q^2)r^2$,

also $\quad y = 8p^3q^3(p^2-q^2)^3(p^2+4pq+q^2)^2r^4$;

therefore $\quad x = 16p^4q^4(p^2-q^2)^4(p^2+q^2)(p^2+4pq+q^2)^3r^6$.

The three numbers will be found to be

$$256r^{12}p^8q^8(p^2-q^2)^8(p^2+4pq+q^2)^6\left\{(p^2+q^2)^2\pm 2pq(p^2-q^2)\right\},$$

and $$256r^{12}p^8q^8(p^2-q^2)^8(p^2+4pq+q^2)^6(p^2+q^2)^2.$$

If we take $p=2$, $q=1$, and $r=\frac{1}{2}$, the three numbers will be

$$2^4.3^8.13^6.13, \qquad 2^4.3^8.13^6.25, \qquad 2^4.3^8.13^6.37.$$

Other answers may be found at pleasure.

[If we take $p=2$, $q=-1$, $r=\frac{1}{6}$, we shall obtain the three numbers 1872, 3600, 5328, which will satisfy the conditions of the question, provided we consider one of the square roots *negative.*]

II. *Solution by* ASHER B. EVANS, M.A.; A. MARTIN; *and others.*

Let x^2-xy+y^2, x^2+y^2, and x^2+xy+y^2 represent three numbers in arithmetical progression. Assume $x=2mn$, and $y=m^2-n^2$; then it will only remain to make

$$m^2+4mn+n^2=\square \ \ldots\ldots\ldots\ldots\ldots(1), \quad \text{and} \quad xy=\text{a cube} \ \ldots\ldots\ldots\ldots\ldots(2).$$

Assume $(m+pn)$ for the root of (1); then we shall have

$$\frac{m}{n}=\frac{(p+1)(p-1)}{2(2-p)}.$$

Let $p=\frac{3}{2}$, then $m=\frac{5}{4}n$; and condition (2) becomes

$$xy=\tfrac{45}{32}n^4=(\tfrac{1}{4}n)^3.2.3^2.5.n=\text{a cube},$$

which condition will be satisfied by taking $n=2^2.3.5^2=300$.

Hence the three numbers are

$$x^2-xy+y^2=1321.3^4.5^8=41797265625,$$
$$x^2+y^2=1681.3^4.5^8=53187890625,$$
$$x^2+xy+y^2=2041.3^4.5^8=64578515625.$$

[These numbers may be obtained from Mr. BILLS's expressions by taking $p=5$, $q=4$, $r^{-1}=3.4.11$.]

3003. (Proposed by Professor SYLVESTER.)—Prove that a uniform circular plate of matter attracting according to the inverse fifth power of the distance will serve to maintain the motion of a particle in the orbit of any circle cutting the circumference of the plate orthogonally, or to speak more strictly, in the portion of such orbit exterior to the plate.

Solution by the REV. R. TOWNSEND, F.R.S.

This very elegant property, though probably admitting of a more direct demonstration, may be inferred without difficulty from the following considerations:—

1. From Newton's general theorem (*Principia*, Book II., Prop. 7, Cor. 2), combined with a well known property of Dr. Salmon (*Modern Geometry*, Vol. I., Art. 179), it appears that a material point P may be made to describe any circle C, by the action of a force F directed to any fixed

centre O in its plane, and varying directly as the distance of P from O, and inversely as the cube of its distance from the polar L of O with respect to C.

2. By direct integration, or otherwise, it may be readily shown that the attraction F of a uniform circular plate C, whose centre is O and whose mass = M, upon any external point P in its plane, for the law of the inverse 5th power of the distance, $= \frac{M.OP}{PT^6}$, where PT is the length of the tangent from P to the circumference of the plate.

3. For any two fixed circles in the same plane, the square of the tangent PT to either from any point P on the other, varies as the distance PL of P from the radical axis L of the circles (*Modern Geometry*, Vol. I., p. 241); which radical axis L is evidently, when the circles intersect at right angles, the polar of the centre O of either with respect to the other.

From these properties combined—the first of which, it may be observed, was shown by the late Sir W. R. Hamilton to be true, not only for the circle, but for any conic as well—that of Professor Sylvester of course immediately follows.

NOTE.—From the known property, that any orbit, capable of being described under the separate actions of two or more different systems of forces, may be described under the united action of all the systems combined,—the square of the velocity under the conjoint action being equal to the sum of the squares of the velocities under the separate actions at every point of the orbit,—it follows that the above property is true, not only for an entire homogeneous plate, with respect to any circle orthogonal to its circumference, but also for a homogeneous ring bounded by any inner and outer circles, with respect to any circle orthogonal to both; and therefore also for a plate or ring which is not homogeneous all through, but composed instead of homogeneous rings of different densities, bounded by circles which, whether finite or infinite in number, are coaxal, with respect to any circle of the entire system orthogonal to them all.

[Professor SYLVESTER remarks that it would be easy to express the law of density on the hypothesis of the Note as a function of the coordinates in the attracting plate, and interesting to do so.]

2963. (Proposed by W. S. BURNSIDE, M.A.)—At the points where a transversal meets a quartic curve four tangents are drawn, which in general meet the curve again in eight points lying on a conic. Now suppose the quartic curve to be reduced to two conics, and the eight-point conic to become two lines: prove that the envelope of the transversal is a curve of the second class, U = 0, and show that the discriminant of U is $\Delta^2\Delta_1^2(\Theta\Theta_1-\Delta\Delta_1)^2$ expressed in terms of the invariants of the two conics.

Solution by the Rev. J. WOLSTENHOLME, M.A.

If the transversal $px+qy+rz=0$ meet the two conics

$$x^2+y^2+z^2=0\ldots\ldots(1),\quad lx^2+my^2+nz^2=0\ldots\ldots(2),$$

the equation of a conic through the four points where the tangents to (1) at points on the transversal meet (2) is

$$(x^2+y^2+z^2)(p^2+q^2+r^2)-(px+qy+rz)^2+k(lx^2+my^2+nz^2)=0\ldots(A),$$

and the equation of a conic through the four points where the tangents to (2) meet (1) is

$$(lx^2+my^2+nz^2)\left(\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}\right)-(px+qy+rz)^2+k'(x^2+y^2+z^2)=0 \ldots\text{(B)}.$$

(A), (B) obviously coincide if $k=\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}$, $k'=p^2+q^2+r^2$, so that the eight points lie on one conic, and the condition that this conic (A) may reduce to two straight lines is

$$(q^2+r^2+kl)(r^2+p^2+km)(p^2+q^2+kn)-2p^2q^2r^2-(q^2+r^2+kl)\,q^2r^2-\ldots\ldots$$
$$\ldots\ldots-\ldots\ldots=0.$$

It is clear that k, k' are factors of the left-hand number, and the other factor is

$$U\equiv p^2mn\,(l+m)\,(l+n)+q^2nl\,(m+n)\,(m+l)+r^2lm\,(n+l)\,(n+m),$$

or the envelope of the transversal, when the conic through the eight points reduces to two straight lines, is a curve of the second class $U=0$, whose discriminant is

$$l^2m^2n^2\,(m+n)^2\,(n+l)^2\,(l+m)^2\equiv\Delta^2\Delta_1^2\,(\Theta\Theta_1-\Delta\Delta_1)^2.$$

2888. (Proposed by S. Roberts, M.A.)—1. If a Cartesian oval has two imaginary axial foci, and consequently two real extra-axial foci, the tangent at any point bisects (internally or externally) the angle between the vector drawn from the real axial focus and the radius of a circle through the extra-axial foci and the point of contact, the radius being drawn thereto.

2. A conic section may be considered as representing a Cartesian oval either with three real axial foci or with two imaginary and one real axial foci. Regarded in the last point of view, what is the property of the conic corresponding to the general case?

Solution by the Proposer.

Take as the equation of the oval, the real axial focus being the pole,

$$\rho^2-2a\rho-2Ax+p^2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and as the equation of the circle, $\rho^2-2Bx+B^2-s^2=0$ .... (2).

We must have, as the condition of passing through the extra-axial foci,

$$\frac{A^2+p^2-a^2}{A}=\frac{B^2+p^2-s^2}{B};$$

hence we get $\rho^2-2a\rho-2Ax+AB+\dfrac{Ba^2-As^2}{B-A}=0.$

Writing ρ', x' for the coordinates of an intersection of (1), (2), we have

$$\rho'=\frac{Ba-As}{B-A},$$

$$2Bx' = \rho'^2 + B^2 - s^2 = \frac{B^2(a^2-s^2) - 2AB(as-s^2) + B(B-A)^2}{(B-A)^2};$$

therefore $$\frac{A\rho'}{\rho'-a} = \frac{Ba-As}{a-s},$$

$$\frac{A\rho'}{\rho-a} - x' = \frac{\left\{2(Ba-As)-(a-s)B\right\}(B-A)^2-(a-s)^2\left\{B(a+s)-2As\right\}}{2(a-s)(B-A)^2}$$

$$= \frac{(B-A)^2-(a-s)^2}{2(a-s)(B-A)^2}\left\{B(a+s)-2As\right\};$$

also $$\frac{A\rho'}{\rho'-a}x' - \rho'^2 = \frac{Ba-As}{a-s}\left\{\frac{\left\{B(a+s)-2As\right\}(a-s)+B(B-A)^2}{2(B-A)^2}\right\} - \left(\frac{Ba-As}{B-A}\right)^2$$

$$= \frac{Ba-As}{2(a-s)(B-A)^2}\left\{(B-A)^2-(a-s)^2\right\}.$$

But X being the intercept of the tangent on the axis, we have

$$\left\{\frac{A\rho'}{\rho'-a} - x'\right\}X = \frac{A\rho'}{\rho'-a}x' - \rho'^2;$$

whence $$X = \frac{Ba-As\,.\,B}{Ba-As+(B-A)S} = \frac{\rho'B}{\rho'+S}, \text{ or } \frac{B-X}{X} = \frac{s}{\rho'}.$$

But B is the distance of the centre of the circle from the origin, therefore the angle between ρ' and s is bisected.

Suppose now we consider an ellipse as representing a special Cartesian oval with two imaginary axial foci; the real focus is at infinity, on the line of the minor axis. We have then the following theorem:—If a circle be drawn through the real foci of an ellipse, the tangent at a point where the circle meets the ellipse will bisect the angle between the radius of the circle drawn to the point and a line through it parallel to the minor axis. In the case of the hyperbola the theorem also holds, but the angle will be bisected externally.

It is usual to say that a conic is a Cartesian oval of which the third focus goes off to infinity. But it must be remembered that a Cartesian is a bicircular quartic of which one focus has gone off to infinity. So that, in the case of conics, there are two real foci at infinity; and we might define a conic as a Cartesian oval axial in two directions. With reference to a circle, we arrive at the curious conclusion that it represents a Cartesian oval of both kinds with reference to any diameter as axis.

It is plain that both species of Cartesians cut orthogonally when triconfocal; and, indeed, the corresponding theorem for bicircular quartics is also true.

2960. (Proposed by W. K. CLIFFORD, B.A.)—The envelope of a series of surfaces of order n, such that two of them can be drawn through an arbitrary point, is a surface of order $2n$, whose equation may be written in

the form $\sqrt{(\alpha X)} + \sqrt{(\beta Y)} + \sqrt{(\gamma Z)} = 0$, where $X = 0$, $Y = 0$, $Z = 0$ are equations of any three surfaces of the series.

The envelope of a net of surfaces of order n, such that two of them can be drawn through *two* arbitrary points, is a surface of order $2n$, whose equation referred to any four surfaces of the net is of the same form as the equation of a quadric referred to four tangent planes.

Solution by R. W. GENESE.

The general form of series will be $k^2\phi + k\psi + \chi = 0$ ....(1),

where ϕ, ψ, χ are *known* functions of nth degree (for two such surfaces may be found to pass through any given point). Let any three of the series be

$$k_1^2\phi + k_1^2\psi + \chi \equiv X, \quad k_2^2\phi + k_2^2\psi + \chi \equiv Y, \quad k_3^2\phi + k_3^2\psi + \chi \equiv Z;$$

thus (1) may be written $\begin{vmatrix} k^2 & k & 1 & 0 \\ k_1^2 & k_1 & 1 & X \\ k_2^2 & k_2 & 1 & Y \\ k_3^2 & k_3 & 1 & Z \end{vmatrix} = 0,$

or $$X(k-k_2)(k-k_3)(k_2-k_3) + \ldots\ldots = 0.$$

If we write this $lX + mY + nZ = 0$ ....(2),

it is easy to see that $\dfrac{(k_2-k_3)^2}{l} + \dfrac{(k_3-k_1)^2}{m} + \dfrac{(k_1-k_2)^2}{n} = 0.$

This shows, by analogy with the well-known trilinear formulæ, that (2) touches $(k_2-k_3)\sqrt{X} + (k_3-k_1)\sqrt{Y} + (k_1-k_2)\sqrt{Z} = 0.$

2995. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—From the top of a tower of height h, particles are projected in all directions in space, with a velocity due to a fall through a height h; show that the mean value of the range is given by the expression

$$2h\int_0^1 (1-x^4)^{\frac{1}{2}}\,dx.$$

Solution by the Rev. T. J. SANDERSON, M.A.; A. MARTIN; S. WATSON; R. W. GENESE; *and others.*

The equation to the trajectory, taking the point of projection as origin and the axis of x parallel to the ground plane, is

$$y = x\tan\alpha - \frac{g}{2V^2\cos^2\alpha}x^2 = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha} \quad (\text{since } V^2 = 2gh).$$

To get the range upon the ground plane, put $y = -h$; then, solving the quadratic, we get $x = 2h\cos\alpha\left\{\sin\alpha \pm (1+\sin^2\alpha)^{\frac{1}{2}}\right\}.$

These two values of x answer, the one to projection upwards, the other to projection in the opposite direction downwards; and the mean of this pair of values is $2h\cos\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}$. Hence the mean value of all possible ranges

$$=\frac{2h\Sigma\left\{\cos\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\right\}}{n}=\frac{2h\Sigma\left\{\cos\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\right\}a^2\cos\alpha\,d\alpha\,d\phi}{na^2\cos\alpha\,d\alpha\,d\phi}$$

(multiplying numerator and denominator by element of surface of sphere)

$$=\frac{2a^2h\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\cos^2\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\,d\alpha\,d\phi}{\text{surface of hemisphere}}=\frac{2a^2h\,.\,2\pi\int_0^{\frac{1}{2}\pi}\cos^2\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\,d\alpha}{2\pi a^2}$$

$$=2h\int_0^{\frac{1}{2}\pi}\cos^2\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\,d\alpha=\text{(putting x for $\sin\alpha$)}\ 2h\int_0^1(1-x^4)^{\frac{1}{2}}\,dx.$$

2767. (Proposed by G. A. Ogilvie.) — If $1+a_1r+a_2r^2+a_3r^3+\ldots$ represents a series of which a_1, a_2, a_3, &c., are the 2nd, 3rd, 4th, &c., terms of the pth order of figurate numbers; show that the sum to n terms of this series is $\dfrac{1-r^n r_p^{-n}}{(1-r)^p}$, where r_p^{-n} denotes the sum of the first p terms in the expansion of r^{-n}.

Solution by the Proposer.

Let $\qquad S=1+a_1r+a_2r^2+a_3r^3+\ldots\ldots+a_{n-1}r^{n-1};$

therefore $\qquad rS=r+a_1r^2+a_2r^3+\ldots\ldots+a_{n-2}r^{n-1}+a_{n-1}r^n;$

therefore $(1-r)S=1+(a_1-1)r+(a_2-a_1)r^2+(a_3-a_2)r^3+\ldots\ldots$
$\qquad\ldots\ldots+(a_{n-1}-a_{n-2})r^{n-1}-a_{n-1}r^n;$

therefore $(1-r)S=1+b_1r+b_2r^2+b_3r^3+\ldots\ldots+b_{n-1}r^{n-1}-a_{n-1}r^n\ \ldots\ldots(\alpha),$

where $a_s-a_{s-1}=b_s=(s+1)$th term in $(p-1)$th order of figurate numbers.

Again, multiplying (α) by r, and subtracting, we have

$(1-r)^2S=1+c_1r+c_2r^2+c_3r^3+\ldots+c_{n-1}r^{n-1}-b_{n-1}r^n-a_{n-1}r^n(1-r)\ \ldots(\beta),$

where $b_s-b_{s-1}=c_s=(s+1)$th term in $(p-2)$th order of figurate numbers.

Multiplying (β) by r, and subtracting, we have

$$(1-r)^3S=1+(c_1-1)r+(c_2-c_1)r^2+(c_3-c_2)r^3+\ldots\ldots$$
$$-c_{n-1}r^n-b_{n-1}r^n(1-r)-a_{n-1}r^n(1-r)^2\ \ldots\ldots\ldots(\gamma).$$

Now it is evident from this series that as soon as $1=c_1=c_2=c_3=\ldots=c_{n-1}$,

then all the terms involving c, except the last, will vanish; and this will happen after p operations, since it is of the pth order.

Hence the series becomes

$$(1-r)^p\,S = 1 - r^n\left\{1 + n(1-r) + \frac{n\,.\,(n+1)}{1\,.\,2}(1-r)^2 + \dots \text{ to } p \text{ terms}\right\} \dots (\delta),$$

for the lowest coefficient of r^n must be 1, and the lowest coefficient but one of r^n must be $n(1-r)$; and, generally, a_n = sum of n terms of pth order

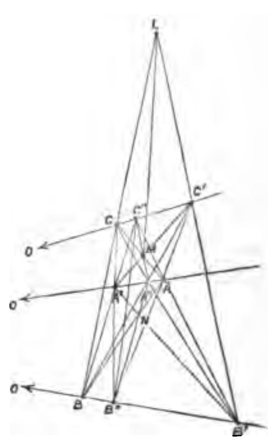
$$= \frac{n\,.\,(n-1)\dots(n+p-1)}{1\,.\,2\dots r}.$$

The series (δ) can be immediately put in the requisite form

$$(1-r)^p\,S = 1 - r^n r^{-n};\quad \text{therefore } S = \frac{1-r^n r_p^{-n}}{(1-r)^p}.$$

2742. (Proposed by F. D. Thomson, M.A.)—Prove that if tangents be drawn to a cubic from the real points of inflexion, the 12 points of contact lie by *threes* on 19 straight lines.

Solution by the Proposer.



1. Consider the figure formed by two copolar triangles ABC, A′B′C′ such that AA′, BB′, CC′ meet in O, and the corresponding sides of the triangles in L, M, N.

Then it is known that LMN is a straight line.

Now a cubic can be drawn through A, B, C, A′, B′, C′, L, M, N: suppose then that the figure is such that L, M, N are the real points of inflexion of the cubic.

Then since LBC and LNM are straight lines, the satellites of LL, BN, CM are in a straight line.

Therefore L, A, A are in a straight line; or, the tangent at A passes through L.

Similarly,	"	A′	"	L,
"	"	B, B′	"	M,
"	"	C, C′	"	N.

Now let A″, B″, C″ be the satellite of AA′, BB′, CC′; then, since the polar conic of a point of inflexion consists of the tangent at the point of inflexion, and another straight line, the tangents at A″B″C″ pass through L, M, N respectively.

Again, since BB′B″ and CC′C″ are straight lines, the satellites of BC, B′C′, B″C″ are in a straight line; therefore L, L, and the satellite of B″C″ are in a straight line; therefore B″C″ passes through L, since L is a point of inflexion. Similarly C″A″ goes through M and A″B″ through N.

2. Again, since AA′A″ and LMN are straight lines, therefore the satellites of AL, A′M, A″N are in a straight line; therefore AC′B″ are in a straight line. Similarly, it can be shown that AC″B′, A′B″C, A′BC″, A″CB′, A″C′B are straight lines.

Hence we have the theorem:—"A chord joining any two of the 12 points of contact of tangents to a cubic which pass through the real points of inflexion passes through a third point of contact." Or, "the 12 points of contact of tangents through the real points of inflexion lie by *threes* on 19 straight lines."

3. It follows from the figure that the pencils $O\{BACL\}$, $O\{ABCM\}$, $O\{ACBN\}$ are harmonic. It is also easily proved analytically from the equation $xyz = (ax+by+cz)^3$, that if PQR be the triangle formed by the tangents at L, M, N, and p, q, r be the points where OA, OB, OC meet the corresponding sides of PQR, then qr passes through L, rp through M, pq through N, and the pencils $P\{QARL\}$ &c. are harmonic.

2942. (Proposed by W. K. Clifford, B.A.)—Let p, q be the foci, and P, Q the asymptotes of a conic; θ the angle it subtends at a point a, and [A] the chord it cuts off from a line A. Then

1. If a line B is drawn through the point a meeting the conic in l, m,

$$al \,.\, am \,.\, \sin BP \sin BQ = \frac{ap^2 \,.\, aq^2 \,.\, \sin^2\theta}{pq^2}.$$

2. If from a point b on the line A tangents L, M are drawn to the conic,

$$\sin AL \sin AM \,.\, bp \,.\, bq = \frac{\sin^2 AP \sin^2 AQ \,.\, [A]^2}{\sin^2 PQ}.$$

(Here al means the distance between the points a, l, and BP means the angle between the lines B, P.)

3. Find analogous propositions for a curve of any order on a plane or on a sphere.

Solution (1) *by the* Rev. J. Wolstenholme, M.A.; (2) *by* J. J. Walker, M.A.

1. If the coordinates of the point a be (X, Y), and the equation of the conic $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then the equation, giving the angle ($\tan^{-1}m$) which a tangent through (X, Y) makes with the axis of x, is

$$(X^2-a^2)\,m^2-2XYm+Y^2-b^2=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (1);$$

whence
$$\tan^2\theta = \frac{4X^2Y^2-4\,(X^2-a^2)\,(Y^2-b^2)}{(X^2-a^2+Y^2-b^2)^2},$$

and
$$\sin^2\theta = \frac{4\,(b^2X^2+a^2Y^2-a^2b^2)}{(X^2+Y^2)^2-2\,(a^2-b^2)\,(X^2-Y^2)+(a^2-b^2)^2};$$

also $$ap^2 \,.\, aq^2 = \left\{(\mathrm{X}-ae)^2+\mathrm{Y}^2\right\}\left\{(\mathrm{X}+ae)^2+\mathrm{Y}^2\right\}$$
$$= (\mathrm{X}^2+\mathrm{Y}^2+a^2-b^2)^2-4\,(a^2-b^2)\,\mathrm{X}^2$$
$$= (\mathrm{X}^2+\mathrm{Y}^2)^2-2\,(a^2-b^2)\,(\mathrm{X}^2-\mathrm{Y}^2)+(a^2-b^2)^2,$$

or $$ap^2 \,.\, aq^2 \,.\, \sin^2\theta = 4\,(b^2\mathrm{X}^2+a^2\mathrm{Y}^2-a^2b^2) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Now if ϕ be the angle which the line B makes with the axis of x, its equation will be $$\frac{x-\mathrm{X}}{\cos\phi} = \frac{y-\mathrm{Y}}{\sin\phi} = r,$$

and for the points where it meets the conic,
$$b^2\,(\mathrm{X}+r\cos\phi)^2+a^2\,(\mathrm{Y}+r\sin\phi)^2 = a^2b^2,$$

whence $$al \,.\, am = \frac{b^2\mathrm{X}^2+a^2\mathrm{Y}^2-a^2b^2}{b^2\cos^2\phi+a^2\sin^2\phi} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3);$$

and, putting i for $\sqrt{(-1)}$, the equations of the asymptotes being

$$y = \pm\frac{b}{a}ix, \text{ we have } \sin^2\mathrm{BP} = \frac{\left(\tan\phi\pm\frac{b}{a}i\right)^2}{(1+\tan^2\phi)\left(1-\frac{b^2}{a^2}\right)},$$

or $$\sin\mathrm{BP}\,.\,\sin\mathrm{BQ} = \frac{\tan^2\phi+\frac{b^2}{a^2}}{(1+\tan^2\phi)\left(1-\frac{b^2}{a^2}\right)} = \frac{4\,(a^2\sin^2\phi+b^2\cos^2\phi)}{pq^2} \;\ldots (4);$$

or, finally, $$\frac{ap^2\,.\,aq^2\,.\,\sin^2\theta}{pq^2} = al\,.\,am\,.\,\sin\mathrm{BP}\,\sin\mathrm{BQ}.$$

2. This second case may be proved at once by means of the formulæ supplied in the solution of the first case, supplemented by the following. (See Quest. 3025.) Let three lines in the same plane make with any axis which they meet angles $\tan^{-1}m_1$, $\tan^{-1}m_2$, and ϕ, respectively, m_1 and m_2 being given by the quadratic $\alpha m^2+\beta m+\gamma = 0$; then the product of the sines of the angles which the first two lines make with the third will be given by the formula $\dfrac{\alpha\sin^2\phi+\beta\sin\phi\cos\phi+\gamma\cos^2\phi}{\left\{(\alpha-\gamma)^2+\beta^2\right\}^{\frac{1}{2}}}$.

From this and equation (1), we have
$$\sin\mathrm{AL}\,.\,\sin\mathrm{AM} = \frac{a^2\sin^2\phi+b^2\cos^2\phi-(\mathrm{X}\sin\phi-\mathrm{Y}\cos\phi)^2}{\left\{(\mathrm{X}^2+\mathrm{Y}^2)-2\,(a^2-b^2)\,(\mathrm{X}^2-\mathrm{Y}^2)+(a^2-b^2)^2\right\}^{\frac{1}{2}}},$$

or $bp\,.\,bq\,.\,\sin\mathrm{AL}\,.\,\sin\mathrm{AM} = a^2\sin^2\phi+b^2\cos^2\phi-(\mathrm{X}\sin\phi-\mathrm{Y}\cos\phi)^2 \ldots (5)$.

Again,
$$[\mathrm{A}]^2 = 4\,\frac{(b^2\mathrm{X}\cos\phi+a^2\mathrm{Y}\sin\phi)^2-(b^2\mathrm{X}^2+a^2\mathrm{Y}^2-a^2b^2)\,(a^2\sin^2\phi+b^2\cos^2\phi)}{(a^2\sin^2\phi+b^2\cos^2\phi)^2}$$
$$= \frac{4a^2b^2}{(a^2\sin^2\phi+b^2\cos^2\phi)}\left\{a^2\sin^2\phi+b^2\cos^2\phi-(\mathrm{X}\sin\phi-\mathrm{Y}\cos\phi)^2\right\} \ldots (6).$$

Now $$\sin^2\mathrm{PQ} = \frac{4a^2b^2}{(a^2-b^2)^2},$$

and, by (4), $\qquad \sin \mathrm{AP} \,.\, \sin \mathrm{AQ} = \dfrac{a^2 \sin^2 \phi + b^2 \cos^2 \phi}{a^2 - b^2};$

whence $\qquad \dfrac{4a^2b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^2} = \dfrac{\sin^2 \mathrm{PQ}}{\sin^2 \mathrm{AP} \,.\, \sin^2 \mathrm{AQ}};$

and, by (6), $a^2 \sin^2 \phi + b^2 \cos^2 \phi - (\mathrm{X} \sin \phi - \mathrm{Y} \cos \phi)^2 = \dfrac{\sin^2 \mathrm{AP} \,.\, \sin^2 \mathrm{AQ}\, [\mathrm{A}]^2}{\sin^2 \mathrm{PQ}}.$

Comparing this result with (5), the equality given in Case 2 of the Question is proved.

[3. Mr. Clifford remarks that "the extensions for a plane are

(curve of class n) ...... $al \,.\, am \,.\, an \ldots \sin \mathrm{BP} \,.\, \sin \mathrm{BQ} \ldots$

$$= \frac{(ap \,.\, aq \,.\, ar \ldots)^{2(n-1)} \sin^2 \mathrm{LM} \,.\, \sin^2 \mathrm{LN} \ldots}{pq^2 \,.\, pr^2 \,.\, qr^2},$$

(curve of order m) ...... $\sin \mathrm{AL} \,.\, \sin \mathrm{AM} \ldots bp \,.\, bq \ldots$

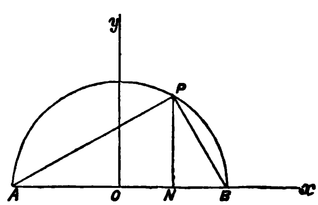
$$= \frac{(\sin \mathrm{AP} \,.\, \sin \mathrm{AQ} \ldots)^{2(m-1)}\, lm^2 \,.\, ln^2 \,.\, mn^2 \ldots}{\sin^2 \mathrm{PQ} \,.\, \sin^2 \mathrm{QR} \,.\, \sin^2 \mathrm{QR}}.$$

where P, Q, R ... are the asymptotes, and p, q, r ... the real foci. These give me ideas of the 'distance' of a point from a line or surface, and they may be extended so as to give the distance of two curves from one another."]

2993. (Proposed by A. Martin.)—Required the average area of all the right-angled triangles whose hypotenuses are equal to $2a$.

Solution by the Rev. T. J. Sanderson, M.A.; R. W. Genese; *the* Proposer; *and many others.*

The vertices of all the triangles will lie upon a semicircle on the hypotenuse AB.



1. Suppose these vertices distributed in proportion to the arc of this semicircle. Then, if there be n triangles, their average area

$$= \frac{a\Sigma(\mathrm{PN})}{n} = \frac{a\Sigma(\mathrm{PN})\,\delta s}{n\delta s}$$

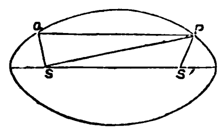
$$= (\text{in limit})\ \frac{a}{\pi a}\int_0^{a\pi} y\,\delta s = \frac{1}{\pi}\int_0^{\pi} a^2 \sin\theta\, d\theta = \frac{2a^2}{\pi}.$$

2. Suppose the vertices so distributed that the abscissæ ON &c. increase uniformly. Then the average area of n triangles so described

$$= \frac{a\Sigma(\mathrm{PN})}{n} = \frac{a\Sigma(\mathrm{PN})\,\delta x}{n\delta x} = \frac{a}{a}\int_0^{a} y\,dx = \frac{\pi a^2}{4}.$$

2964. (Proposed by R. W. GENESE.) — The average focal radius-vector of (1) an ellipse, (2) a prolate spheroid, is half the major axis, the distribution being proportional in the former to the arc, in the latter to the surface.

I. *Solution by the* REV. T. J. SANDERSON, M.A.



1. Take P any point in the ellipse, and draw PQ parallel to the major axis.

Now, from the nature of the distribution, it is clear that for every radius-vector SP, drawn to a point P on one side of the minor axis, there corresponds another radius-vector SQ drawn to a corresponding point Q on the other side of the minor axis. And the mean length of this pair of radii-vectores

$$=\tfrac{1}{2}(SP+SQ)=\tfrac{1}{2}(SP+S'P)=\text{half the major axis},$$

and so of every other such pair. Therefore the mean length of *all* the radii-vectores drawn from the focus is equal to half the major axis.

2. For the prolate spheroid, likewise, all the focal radii-vectores may be divided into pairs, such as SP, SQ, whose mean is half the major axis; and therefore the mean of *all* is again equal to half the major axis.

NOTE.—An exactly similar proof applies to show that if the distribution of the radii-vectores is such that the abscissæ of their extremities increase uniformly, their mean length is still equal to half the major axis.

If the radii are distributed at equal angular intervals, their mean will be found equal to half the *minor* axis. (See Todhunter's *Integral Calculus*, Ch. XIV., Ex. 5.)

II. *Solution by the* PROPOSER.

1. If x be measured from the centre, we have, in the ellipse,

$$\text{Mean value}=\frac{\int SP\,.\,ds}{\int ds}=\frac{\int(a+ex)\,ds}{\int ds}=a+e\frac{\int x\,ds}{\int ds};$$

and clearly $\int x\,ds$ over the semi-ellipse $=0$.

2. In the spheroid, taking as element of surface $2\pi y\,dx$, we have

$$\text{Mean value}=\frac{\int(a+ex)\,2\pi y\,dx}{\int 2\pi y\,dx}=a+e\frac{\int xy\,dx}{\int y\,dx};$$

and we may regard the second term of this expression as $e\times$ (the mean of x) $=0$. It may be useful to notice that, in one integral, $y\,dx$ is always positive here.

III. *Solution by* ASHER B. EVANS, M.A.

Let n be the number of positions of P; then, since $SP+S'P=2a$, we have

$$\frac{\Sigma(SP)+\Sigma(S'P)}{n}=2a\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But it is evident that $\Sigma\,(SP) = \Sigma\,(S'P)$ for the entire ellipse; therefore, from (1),
$$\frac{2\Sigma\,(AP)}{n} = 2a;$$
or the mean value of AP is a.

A similar method applies to the spheroid.

2617. (Proposed by R. TUCKER, M.A.)—Solve the differential equation
$$x^2+y^2+\frac{2\,(1+p^2)}{q}\,(y-xp) = k^2,$$
where $p = \dfrac{dy}{dx}$, $q = \dfrac{d^2y}{dx^2}$, and k is a constant.

I. *Solution by the* PROPOSER, *and others.*

Changing to polar coordinates, we get
$$r^3\left(r+\frac{d^2r}{d\theta^2}\right) = k^2\left(r^2+2\,\frac{dx^2}{d\theta^2}-r\,\frac{d^2r}{d\theta^2}\right);$$
and then writing u for $\dfrac{1}{r}$, we have
$$\cdot\;\frac{1-k^2u^2}{u\,(1+k^2u^2)}\,\frac{d^2u}{d\theta^2} = 1+\frac{2}{u^2\,(1+k^2u^2)}\,\frac{du^2}{d\theta^2},$$
an answer to which is $k^2u^2+(A\cos\theta+B\sin\theta)\,u+1 = 0$,
which is the equation to a circle.

The differential equation arises from the question, "To find the curve such that the tangent to the circle of curvature at every point from a fixed point may be constant."

II. *Solution by* J. J. WALKER, M.A.; W. H. LAVERTY, B.A.; *and others.*

Writing the given equation
$$(x^2+y^2-k^2)\,q = 2\,(1+p^2)\,(xp-y) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
differentiating again, and writing r for $\dfrac{d^3y}{dx^3}$, there results
$$(x^2+y^2-k^2)\,r = 6pq\,(xp-y) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$
Dividing (2) by (1), $$\frac{r}{q} = \frac{3pq}{1+p^2}.$$
Multiplying by dx and integrating,
$$\log cq = \tfrac{3}{2}\log(1+p^2),\quad\text{or}\quad cq = (1+p^2)^{\frac{3}{2}}\quad\text{or}\quad \frac{cq}{(1+p^2)^{\frac{3}{2}}} = 1.$$
Multiplying by pdx and again integrating,
$$\frac{c}{(1+p^2)^{\frac{1}{2}}} = y+c',\quad\text{whence}\quad p = \frac{\pm\left\{c^2-(y+c')^2\right\}^{\frac{1}{2}}}{y+c'}.$$

Replacing now p by $\frac{dy}{dx}$, inverting, and multiplying by dy,

$$dx = \pm \frac{(y+c')\,dy}{\left\{c^2-(y+c')^2\right\}^{\frac{1}{2}}}.$$

Integrating a third time, $x+c'' = \mp\left\{c^2-(y+c')^2\right\}^{\frac{1}{2}}$.

Squaring and arranging, $x^2+y^2+2c''x+2c'y = c^2$,

writing the single arbitrary c^2 instead of $c^2-c'^2-c''^2$. If, in the given equation, k^2 stands for any arbitrary constant, the above is its complete primitive; otherwise its primitive is

$$x^2+y^2-2c''x+2c'y = k^2,$$

as appears by differentiating twice, eliminating c' and c'', and identifying the result with the proposed equation.

2981. (Proposed by the Rev. G. H. Hopkins, M.A.)—If the sum of the squares of two consecutive integers be equal to the square of another integer, find their general values, and show how to find any number of particular solutions.

I. *Solution by* Morgan Jenkins, M.A.

The general solution of $u^2+v^2=\square$ is $u=m(a^2-b^2)$, $v=m(2ab)$, where a, b are any integers prime to each other, and m is any integer, or, if $a-b$ be even, half any integer.

In this case, therefore, $m=1$ or $\frac{1}{2}$; whence

$$a^2-b^2-2ab = \pm 1 \text{ or } \pm 2, \quad i.e., \quad (a-b)^2-2b^2 = \pm 1 \text{ or } \pm 2.$$

The general solution of $(a-b)^2-2b^2 = \pm 1$ is, by the application of a theorem of Lagrange's, $\quad a-b=p_x, \quad b=q_x,$

where $\frac{p_x}{q_x}$ is any xth convergent to $\sqrt{2}$ and $p_x^2-2q_x^2=+1$ and -1 alternately. (We cannot have $a-b=-p_x$, because p_x is $>q_x$.)

This gives $\quad u=p_x(p_x+2q_x), \quad v=2q_x(p_x+q_x),$

that is, $\quad u=p_xp_{x+1}, \quad v=2q_xq_{x+1}.$

When the values ± 2 are taken instead of ± 1, we shall only have the values of u and v interchanged. From the differential equation obtained from the law of formation of successive convergents, we have

$$p_x = \tfrac{1}{2}\left\{(1+\sqrt{2})^x+(1-\sqrt{2})^x\right\}, \quad q_x = \frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^x-(1-\sqrt{2})^x\right\};$$

and the pairs of integers are (3, 4), (21, 20), (119, 120), &c.

II. *Solution by* ASHER B. EVANS, M.A.; R. W. GENESE; *and others.*

Put $c = a$ and $d = b$ in the identity

$$(a^2+b^2)(c^2+b^2) = (ac-bd)^2+(ab+cd)^2,$$

and it becomes $\quad (a^2+b^2)^2 = (a^2-b^2)^2+(2ab)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$

The quantities a and b must be so chosen in (1) as to satisfy in integers

$$2ab-(a^2-b^2) = \pm 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (2) we find $\quad b+a = \sqrt{(2a^2 \pm 1)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$

hence $\quad x^2-2a^2 = \pm 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

must be satisfied in integers. To obtain all the integral values of x and a that will satisfy (4), extract the square root of 2 according to the method of continued fractions, and the terms of each approximate fraction will give one set of values for x and a; thus

$$\frac{x_1}{a_1} = \frac{1}{1}, \quad \frac{x_2}{a_2} = \frac{3}{2}, \quad \frac{x_3}{a_3} = \frac{7}{5}, \quad \frac{x_4}{a_4} = \frac{17}{12}, \quad \&c.$$

But $\quad b = x-a; \quad$ therefore $b_1 = 0, \; b_2 = 1, \; b_3 = 2, \; \&c.$

When we have two consecutive values of a and b, other values can readily be found, for

$$\left.\begin{aligned} a_n &= 2a_{n-1}+a_{n-1} \\ b_n &= 2b_{n-1}+b_{n-1} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The general values of a and b are

$$a = \mathrm{C}(n,1)+2\mathrm{C}(n,3)+(2^2)\mathrm{C}(n,5)+(2^3)\mathrm{C}(n,7)+\&c. \ldots\ldots(6),$$

$$b = \mathrm{C}(n-1,1)+2\mathrm{C}(n-1,3)+(2^2)\mathrm{C}(n-1,5)+(2^3)\mathrm{C}(n-1,7)+\&c.\ldots(7);$$

where $\quad \mathrm{C}(n,r) = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots r}.$

By taking n equal to 1, 2, 3, 4, &c., we find, from (1), (6), and (7), all the solutions to the question.

If $n=1$, $\quad a=1, \; b=0$, and $1^2+0^0 = 1^0$;

$n=2$, $\quad a=2, \; b=1$, and $3^2+4^2 = 5^2$;

$n=3$, $\quad a=5, \; b=2$, and $21^2+20^2 = 29^2$;

$n=4$, $\quad a=12, \; b=5$, and $119^2+120^2 = 169^2$;

&c. &c. &c.

III. *Solution by the* PROPOSER.

Let x, $x-1$, and y be such integers; then $x^2+(x-1)^2 = y^2$. This equation can be thrown into the form $(2x-1)^2-2y^2 = -1$. By Todhunter's *Algebra*, Art. 639, we have

$$y = \frac{2}{2\sqrt{2}}\left\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\right\}, \quad x = \frac{1}{2}\left[1+\tfrac{1}{2}\left\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\right\}\right] \quad\ldots\ldots(A),$$

n being any odd integer, and $x=1$, $y=1$ a particular solution. The values of x, y thus obtained are the general values required; other particular solutions, as 4, 5, 21, 29, can be found by putting $n = 3, 5$. The process of expansion becomes very tedious for higher values of n than 5; to obtain,

then, other values of x, y, certain properties of these numbers, easily obtainable from (A), must be employed.

Let x_1, y_1; x_3, y_3; x_5, y_5 ...... $x_{2n'+1}, y_{2n'+1}$ be the values of x, y, when n is put equal to 1, 3, 5 ...... $2n'+1$ in the expressions (A). Then it can be easily proved that

$$y_{2p+1}+y_{2p-1} = 2(x_{2p+1}-x_{2p-1}) \text{ ....(B).}$$

But $(x_{2p+1}-1)^2+x^2_{2p+1} = y^2_{2p+1}$, and $(x_{2p-1}-1)^2+x^2_{2p-1} = y^2_{2p-1}$;

therefore $2(x_{2p+1}-x_{2p-1})(x_{2p+1}-x_{2p-1}-1) = (y_{2p+1}-y_{2p-1})(y_{2p+1}+y_{2p-1})$;

therefore we must have $y_{2p+1}-y_{2p-1} = x_{2p+1}+x_{2p+1}-1$ ....(C).

From (C) + (B) and (C) − (B) we obtain

$$2y_{2p+1} = 3x_{2p+1}-x_{2p-1}-1, \quad 2y_{2p-1} = x_{2p+1}-3x_{2p-1}+1 \text{ ...... (1, 2).}$$

From (1), $2y_{2p-1} = 3x_{2p-1}-x_{2p-3}-1$, $\therefore\ 4y_{2p-1} = x_{2p+1}-x_{2p-3}$ ... (3, 4).

From (2) and (3), by subtraction,

$$x_{2p+1} = 6x_{2p-1}-x_{2p-3}-2 \text{ .... (5).}$$

From (4) and (5),

$$4y_{2p-1} = 6x_{2p-1}-2x_{2p-3}-2, \quad x_{2p-1} = 6x_{2p-3}-x_{2p-5}-2 \text{ .... (6, 7).}$$

From (6) and (7), $4y_{2p-1} = 34x_{2p-3}-6x_{2p-5}-14$;

therefore $y_{2p-1} = \frac{1}{4}(17x_{2p-3}-3x_{2p-5}-7)$, and $x_{2p-1} = 6x_{2p-3}-x_{2p-5}-2$.

x_{2p-1}	$= 6x_{2p-3}-x_{2p-5}-2$		y_{2p-1}	$= \frac{1}{4}(17x_{2p-3}-3x_{2p-5}-7)$	
x_1	=	1	y_1	=	1
x_3	=	4	y_3	=	5
x_5	= 24−1−2 =	21	y_5	= $\frac{1}{4}$(68−3−7) =	29
x_7	= 126−4−2 =	120	y_7	= $\frac{1}{4}$(357−12−7) =	169
x_9	= 720−21−2 =	697	y_9	= $\frac{1}{4}$(2040−63−7) =	985
x_{11}	= 4182−120−2 =	4060	y_{11}	= $\frac{1}{4}$(11849−360−7) =	5741
x_{13}	= 24360−697−2 =	23661	y_{13}	= $\frac{1}{4}$(69020−2091−7) =	33461

2972. (Proposed by S. Watson.)—OB and OC are any two semi-diameters of an ellipse conjugate to each other; find (1) the locus, and (2) the area, of the curve which is the intersection of normals at B and C.

Solution by R. W. Genese; R. Tucker, M.A.; A. Martin; Rev. J. Wolstenholme, M.A.; *and others.*

1. If θ, $\theta+\frac{1}{2}\pi$ be the eccentric angles of B, C, and $c^2 = a^2-b^2$, the equations of the normals are

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta} = c^2, \quad \frac{ax}{\sin\theta}+\frac{by}{\cos\theta} = -c^2;$$

hence, eliminating θ, we readily find the equation of the locus to be

$$2(a^2x^2+b^2y^2)^3 = c^4(a^2x^2-b^2y^2)^2,$$

which is a curve consisting of four loops of equal area.

2. Area $= \iint dx\,dy = \dfrac{1}{ab}\iint dx'dy'$, where $x' = ax$, $y' = by$,

and therefore $$2(x'^2+y'^2)^3 = c^4(x'^2-y'^2)^2;$$

or, putting $x' = r\cos\theta$, $y' = r\sin\theta$, we have $r^2 = \frac{1}{2}c^4\cos^2 2\theta$,

$$\text{Area} = \frac{8}{2ab}\int^{\frac{1}{4}\pi} r^2 d\theta = \frac{2}{ab}\int_0^{\frac{1}{4}\pi} c^4\cos^2 2\theta\, d\theta = \frac{\pi c^4}{4ab}.$$

Otherwise:

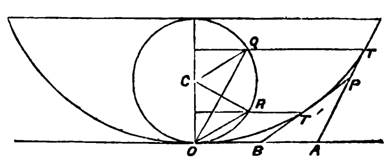
$$\text{Area} = c^4\int_0^{\frac{1}{2}\pi}\frac{(a^2\cos^2\theta-b^2\sin^2\theta)^2\,d\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^3} = c^4\int_0^{\infty}\frac{(a^2-b^2z^2)^2\,dz}{(a^2+b^2z^2)^3}$$

$$= c^4\left\{4a^4\int_0^{\infty}\frac{dz}{(a^2+b^2z^2)^3} - 4a^2\int_0^{\infty}\frac{dz}{(a^2+b^2z^2)^2} + \int_0^{\infty}\frac{dz}{a^2+b^2z^2}\right\}$$

$$= c^4\left\{\frac{3\pi}{4ab} - \frac{\pi}{ab} + \frac{\pi}{2ab}\right\} = \frac{\pi c^4}{4ab}.$$

2998. (Proposed by M. W. Crofton, F.R.S.)—If two tangents to a cycloid include a constant angle, show that their sum has a constant ratio to the included arc of the curve.

I. *Solution by* William Roberts, Jun.



Let $PT = t$, $PT' = t'$, arc $TT' = \Sigma$, $\angle QCO = \theta$, $\angle RCO = \theta'$;

then $t = TA - PA$, $t' = PB - BT'$;

therefore $t + t' = TA - T'B + PB - PA$.

But $TA = QO = \frac{1}{2}\text{arc } OT = \frac{1}{2}s$, $T'B = RO = \frac{1}{2}\text{arc } OT' = \frac{1}{2}s'$;

therefore $$TA - T'B = \tfrac{1}{2}\Sigma,$$

also $$(PB-PA)\sin BPA = AB(\sin\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta')$$
$$= (QT-RT')(\sin\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta') = \tfrac{1}{4}(\theta-\theta')(s-s');$$

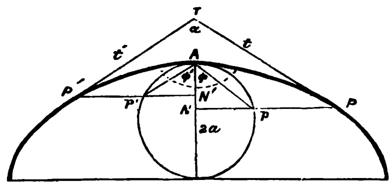
therefore $$PB - PA = \frac{(\theta-\theta')\Sigma}{4\sin BPA} = \frac{BPA\,.\,\Sigma}{2\sin BPA};$$

therefore $$\frac{t+t'}{\Sigma} = \frac{1}{2}\left(1 + \frac{BPA}{\sin BPA}\right),$$

which ratio is constant if the angle BPA is constant.

II. *Solution by* R. W. GENESE.

With the notation of the fi- we have



$$t\cos\phi - t'\cos\phi' = NN'$$
$$= 2a\cos^2\phi - 2a\cos^2\phi'$$
$$= 2a\sin(\phi'-\phi)\sin(\phi'+\phi),$$
$$t\sin\phi + t'\sin\phi' = PN + P'N'$$
$$= 2a\sin(\phi'+\phi)\cos(\phi'-\phi)$$
$$+ 2a(\pi-\phi-\phi');$$

therefore $\quad \dfrac{t}{2a}\sin(\phi'+\phi) = \sin(\phi'+\phi)\cos\phi + (\pi-\phi-\phi')\cos\phi';$

and $\quad \dfrac{t'}{2a}\sin(\phi'+\phi) = \sin(\phi'+\phi)\cos\phi' + (\pi-\phi-\phi')\cos\phi;$

therefore, putting $\phi+\phi' =$ angle between t and $t' = \alpha$,

we have $\quad \dfrac{t+t'}{2a}\sin\alpha = (\sin\alpha + \pi - \alpha)(\cos\phi + \cos\phi');$

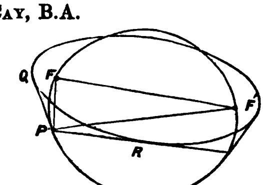
but $\quad$ arc $AP = 2Ap = 4a\cos\phi$, and $AP' = 4a\cos\phi'$,

therefore $\quad \dfrac{t+t'}{\text{arc } PP'} = \dfrac{\sin\alpha + \pi - \alpha}{2\sin\alpha} =$ a constant.

2744. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—A circle passes through the real foci of a conic, prove that the points where the common tangents touch the circle lie on one of the tangents to the conic at the ends of the minor axis.

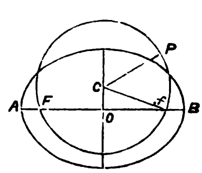
I. *Solution by* W. S. McCAY, B.A.

Draw the tangent PR; then $\angle$ QPF = F'PR, and $\angle$ QPF = FF'P, since QP touches the circle, therefore $\angle$ F'PR = FF'P; and therefore PR is parallel to FF'. Hence the theorem follows.



II. *Solution by* STEPHEN WATSON.

Let O, C be the centres of an ellipse and a circle passing through its foci F, f, and P the point of contact of a common tangent with the circle. Put $OC = m$, $\angle PCO = \frac{1}{2}\pi + \phi$, and take a, b, c as usual. Then the coordinates of P are



$$x_1 = (c^2+m^2)^{\frac{1}{2}}\cos\phi, \text{ and } y_1 = m + (c^2+m^2)^{\frac{1}{2}}\sin\phi \quad \text{.........(1)};$$

also the equation of a tangent to the ellipse perpendicular to CP is

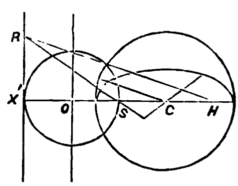
$$x + y\tan\phi = (a^2 + b^2\tan^2\phi)^{\frac{1}{2}} \quad\text{.........................}(2).$$

The condition that this shall pass through (1) gives a quadratic in $\sin\phi$, from which $\quad (c^2+m^2)^{\frac{1}{2}}\sin\phi + m = (a^2-c^2)^{\frac{1}{2}} = b,$

that is, $y_1 = b$, which proves the proposition, as the proof is similar in the hyperbola.

2880. (Proposed by R. Tucker, M.A.)—Given the focus, directrix, and length between the extremities of the major and minor axes of an ellipse; show how to find the major axis and the directions of conjugate diameters by an easy geometrical construction.

Solution by the Proposer.



Let S be the nearer focus, and O the foot of the perpendicular from S on the directrix. Draw a circle with centre O and radius OS. Next describe an orthogonal circle to this, having its centre in OS produced, and radius = the given distance between the extremities of the axes; and let its centre be C: this is the centre of the ellipse. H, the other focus, can now be found; and the tangent to the original circle from H is equal to the major axis. Again, if any point R be taken on a tangent to the first circle where SO cuts it (at X′ suppose) and joined to S and H; then lines through C, parallel to HR and perpendicular to RS respectively, give the directions of conjugate diameters.

2786. (Proposed by J. J. Walker, M.A.)—If ρ be the angular radius of a primary or secondary rainbow, corresponding to angles (θ, θ') of incidence and reflexion at the surface of the rain-drop, and n be the number of internal reflexions, prove that

$$\cos\tfrac{1}{2}(n\pi-\rho) = \cos(\tfrac{1}{2}n\pi-\theta)(n\sin\theta')^{n+1}.$$

Solution by the Proposer.

In the case of the primary bow, it is well known that $\frac{1}{2}\rho = 2\theta' - \theta$, where θ, θ' are determined by the equations

$$\sin\theta = \mu\sin\theta', \quad 2\cos\theta = \mu\cos\theta' \quad\text{....................}(1).$$

Hence $\sin\frac{1}{2}\rho = \sin(2\theta'-\theta) = (2\sin\theta'\cos\theta - \sin\theta\cos\theta')\cos\theta' + \sin\theta\sin^2\theta'.$

The terms within brackets vanish, in virtue of equation (1); hence

$$\sin\tfrac{1}{2}\rho = \cos\tfrac{1}{2}(\pi-\rho) = \sin\theta\sin^2\theta' = \cos(\tfrac{1}{2}\pi-\theta)(\sin\theta')^2.$$

Again, in the case of the secondary bow, $\frac{1}{2}\pi - \frac{1}{2}\rho = 3\theta' - \theta$, where

$$\sin\theta = \mu \sin\theta', \quad 3\cos\theta = \mu\cos\theta' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence $\cos\frac{1}{2}\rho = \sin(3\theta' - \theta)$

$$= 3\sin\theta'\cos\theta - \sin\theta\cos\theta' + 4\sin^2\theta'(\sin\theta\cos\theta' - \sin\theta'\cos\theta).$$

The first two terms vanish, and those within brackets are equal to $2\sin\theta'\cos\theta$, in virtue of equations (2).

Hence $\cos\frac{1}{2}\rho = 8\sin^3\theta'\cos\theta$, or $\cos\frac{1}{2}(2\pi - \rho) = \cos(\frac{1}{2}\pi - \theta)(2\sin\theta')^3$.

The formula does not hold for the tertiary bow.

2668. (Proposed by W. S. Burnside, M.A.)—If the complete solution of the differential equation $\dfrac{d^n y}{dx^n} + f(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \mathrm{F}\left(\dfrac{d^{n-2}y}{dx^{n-2}} \ldots \dfrac{dy}{dx}, y, x\right) = 0$ be $y = \phi(x, x_1, x_2, \ldots x_n)$, where x_1, x_2, &c. are the n arbitrary constants introduced by integration; prove that

$$\begin{vmatrix} \dfrac{d\phi}{dx_1} & \dfrac{d\phi}{dx_2} & \cdots & \dfrac{d\phi}{dx_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{d\phi^{(n)}}{dx_1} & \dfrac{d\phi^{(n)}}{dx_2} & \cdots & \dfrac{d\phi^{(n)}}{dx_n} \end{vmatrix} \epsilon^{\int f(x)\,dx} = \text{constant}, \quad \text{where } \phi^{(r)} = \frac{d^r\phi}{dx^r}.$$

Solution by the Proposer.

First, it is well known that

$$\begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} & \cdots & y_n^{(2)} \\ \vdots & \vdots & \vdots & & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \epsilon^{\int f(x)\,du} = \text{constant},$$

where $y_1, y_2, y_3, \ldots y_n$ are the n particular integrals of the linear differential equation

$$\frac{d^n y}{dx^n} + f(x)\frac{d^{n-1}y}{dx^{n-1}} + f_1(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + f_{n-1}(x)y = 0 \quad \ldots\ldots(1).$$

Secondly, if $y = \phi(x, x_1, x_1 \ldots x_n)$ be the complete solution of the differential equation $\Gamma(x, y, y', y'' \ldots y^{(n)}) = 0$, the complete solution of the linear differential equation

$$\frac{d^n z}{dx^n}\cdot\frac{d\Gamma}{dy^{(n)}}+\frac{d^{n-1}z}{dx^{n-1}}\cdot\frac{d\Gamma}{dy^{n-1}}+\frac{d^{n-2}z}{dx^{n-2}}\cdot\frac{d\Gamma}{dy^{n-2}}+\ \ldots\ldots+z\cdot\frac{d\Gamma}{dy}=0\ldots(2)$$

is
$$z=c_1\frac{d\phi}{dx_1}+c_2\frac{d\phi}{dx_2}+\ \ldots\ldots+c_n\frac{d\phi}{dx_n}.$$

In the present case, where $\Gamma(x, y, y' \ldots y^{(n)}) \equiv y^n + f(x)\, y^{(n-1)} + \ldots$, the linear differential equation (2) becomes

$$\frac{d^n z}{dx^n}+f(x)\frac{d^{n-1}z}{dx^{n-1}}+\ \ldots\ldots+\psi(x)\,.\,z=0.$$

Now, applying the first theorem to this equation, remembering that $\frac{d\phi}{dx_1}, \frac{d\phi}{dx_2}, \ldots\ldots \frac{d\phi}{dx_n}$ take the place of $z_1, z_2, z_3, \ldots z_n$, we have the theorem in the question proved.

2734. (Proposed by J. Wilson.)—Prove that

$$n^{n+1}=(n+1)\,S_n-\frac{(n+1)\,n}{1\,.\,2}\,S_{n-1}+\frac{(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3}\,S_{n-2}-\ldots\pm S_0,$$

where $S_1, S_2, S_3, \ldots S_n$ are the sums of the first, second, third, &c. powers of the terms of the series 1, 2, 3, 4, ... to n terms.

Solution by the Proposer.

We have $\quad S_{n+1}-\frac{n+1}{1}\,S_n+\frac{(n+1)\,n}{1\,.\,2}\,S_{n-1}-\ldots\pm S_0$

$$=(1^{n+1}+2^{n+1}+\ldots n^{n+1})-\frac{n+1}{1}(1^n+2^n+3^n+\ldots n^n)$$
$$+\frac{(n+1)\,n}{1\,.\,2}(1^{n-1}+2^{n-1}+\ldots n^{n-1})-\ldots\pm(1+1+1\ldots\text{to } n \text{ terms})$$
$$=(1-1)^{n+1}+(2-1)^{n+1}+\ldots(n-1)^{n+1}=1^{n+1}+2^{n+1}+\ldots(n-1)^{n+1}.$$

Hence we find

$$S_{n+1}-\left\{1^{n+1}+2^{n+1}+\ldots(n-1)^{n+1}\right\}=n^{n+1}$$
$$=\frac{n+1}{1}\,S_n-\frac{(n+1)\,n}{1\,.\,2}\,S_{n-1}+\ldots\pm S_0=(n).$$

2938. (Proposed by T. Cotterill, M.A.)—Let X, Y, Z be the intersections of the three pairs of lines through four points A, B, C, D. Then a line in their plane contains a pair of points harmonic conjugates to the

conics through ABCD. Show that the locus of such points on the tangents to a curve of the kth class is a curve of the $(3k)$th order, having multiple points of the order k at each of the seven points A, B, C, D, X, Y, Z, the tangents at each of these points being easily found. If the first curve touch a line through two of the points A, B, C, D, the second curve contains this line. If the first curve is a conic through the same four points, the second curve contains the conic.

Solution by the Rev. J. WOLSTENHOLME, M.A.

If the four points A, B, C, D be taken to be $x^2 = y^2 = z^2$, the equation of the series of conics will be $lx^2 + my^2 + nz^2 = 0$, with the condition $l + m + n = 0$; and the two points $(x_1y_1z_1)$, $(x_2y_2z_2)$ will be conjugate to this series of conics, if $x_1x_2 = y_1y_2 = z_1z_2$. Hence on any straight line $px + qy + rz = 0$ the two points given by the equations $px + qy + rz = 0$, $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0$ will be conjugates. If now this straight line be a tangent to a given curve of the kth class, p, q, r are connected by an equation $f(p, q, r) = 0$ of the kth degree; and eliminating, we have for the locus of the points on all such tangents

$$f\left\{x(y^2 - z^2),\ y(z^2 - x^2),\ z(x^2 - y^2)\right\} = 0 \quad \text{.................. (A)}$$

a curve of the $3k$th order for their locus.

At the four points A, B, C, D the equations $px + qy + rz = 0$, $pyz + qzx + rxy = 0$ coincide; and these points are therefore points on the locus for each of the k tangents drawn through the point, or each is a multiple point of the order k. At the point (X, Y, Z) the equation $pyz + qzx + rxy = 0$ is always satisfied; and these are then also multiple points of the order k. The first curve will touch a line through two of the four points A, B, C, D if $f(0, 1, -1) = 0$, and in this case the line $(y = z)$ through the two points satisfies the equation (A). If the first curve be a conic through the four points, the relation between p, q, r is $\frac{p^2}{l} + \frac{q^2}{m} + \frac{r^2}{n} = 0$, with the condition $l + m + n = 0$. In this case the locus is

$$\frac{x^2(y^2 - z^2)^2}{l} + \frac{y^2(z^2 - x^2)^2}{m} + \frac{z^2(x^2 - y^2)^2}{n} = 0;$$

and when this meets $lx^2 + my^2 + nz^2 = 0$, we have

$$-(my^2 + nz^2)(y^2 - z^2)^2 + \frac{y^2}{m}(lz^2 + my^2 + nz^2)^2 + \frac{z^2}{n}(ly^2 + my^2 + nz^2)^2 = 0$$

or
$$-(my^2 + nz^2)(y^2 - z^2)^2 + \frac{y^2}{m}(my^2 - mz^2)^2 + \frac{z^2}{n}(nz^2 - ny^2)^2 = 0,$$

by reason of the condition $l + m + n = 0$. But this equation is identically true, or the locus contains the conic.

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[TURN OVER

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